BEL, BEX, BME, BAME, BIE, BCT

a reference book



G.L. Book House

CONTROL SYSTEM

BACHELOR OF ENGINEERING

T.U./P.U./Po.U.



Salient Features

- Chapterwise Content
- Easy and Lucid Language
- Illustration of Examples
- Solved Numerical



Bachelor in Electronics and communiction engineering
Bachelor in Industrial
Bachelor in Electrical
Bachelor in Automobile
Bachelor in Mechanical
Bachelor in Computer Technology

Er. Umesh Kanta Ghimire (UKG)

Scanned with CamScanner

त्रिभुवन विश्वविद्यालय
Tribhuvan University
इन्जिनियरिङ्ग अध्ययन संस्थान
Institute of Engineering
थापाथली क्याम्पस
THAPATHALI CAMPUS

GPO box- 280, Thapathali, Kathmandu
Tel: 4-246465, 4-246307, Fax: 977-1-4247340
E-mail : tcampus@ioe.edu.npu
गोश्वारा पो.ब.नं.- २८०, थापाथली, काठमाडौं
फोन- ४२४६४६५, ४२४६३०७
फ्याक्स- ९७७-१-४२४७३४०

FOREWORD



Teaching Control System for long time, it's every time required a good reference book for students of **Bachelor degree** for different faculties. It is my pleasure to go through the written **Reference Book** on **Control System** and I found its best for the fulfillment of students need not only the theoretical base but it will be practically usable to design the control systems. The book is well explained in simple language as well as sufficient numerical examples which obviously lead to better achievements in this discipline. It covers entirely from modeling of any system to compensation design.

In this context, I appreciate the efforts of well renowned teacher who has been teaching this subject for last many years to present such difficult subject in simple way which is applicable for all related students.

I wish the author all success and thank for such essential contribution in the field of engineering.

INSTITUTE OF ENGINEERING
THAPATHALI CAMPUS

Er. Shanta Maharjan
Associate Professor
M.Sc. in Electronics Engineering
IOE, Thapathali Campus
1st January 2018

Scanned with CamScanner

CONTENTS

Chapter 1 CONTROL SYSTEM BACKGROUND

1.1 History of control system and its importance 1
1.2 Control system : Characterisics and Basic Features 2
1.3 Types of control system and their comparison 3
1.3.1 Open loop control system (OLCS) 3
1.3.2 Closed loop control system (CLCS) 3
1.3.3 Comparison between open loop and closed loop system 4
1.3.4 Some example of control system 4
1.4 Solved problems 6

Chapter 2 COMPONENT MODELING

2.1 Transfer function notation and differential equations 8
2.1.1 Characteristics equation of a system 9
2.1.2 Transfer function of closed loop control system 10
2.2 Modeling of mechanical components: mass, spring and damper 11
2.2.1 Resisting forces on a translational system 12
2.2.2 Resisting force on a rotational system 12
2.2.3 D' Alembert's principle 13
2.2.4 Review of laplace transform 13
2.2.5 Concepts of mass–less body 18
2.3 Modeling of electrical components: inductance, resistance, capacitance, dc and ac motor, transducers and operational amplifier 21
2.3.1 Resistance 21
2.3.2 Inductance 21
2.3.3 Capacitance 21
2.3.4 DC–motor 25
2.3.4.1 Armature control method for speed control of dc–motor (closed loop control system) 25
2.3.4.2 Field control method for speed control of dc–motor (open loop control system) 27
2.3.5 Operational amplifier (OP–Amp) 28
2.3.6 Transducers 31
2.4 Electrical circuit analogy 31
2.4.1 Force-voltage (F–V) analogy 31
2.4.2 Force-current (F–I) analogy 32

Scanned with CamScanner

2.4.3 Torque voltage analogy: (T–V) analogy 36
2.4.4 Torque current analogy: (T–I) analog 36
2.5 Linearized approximation of non–linear characteristics 38
2.6 Solved problems 41

Chapter 3 SYSTEM TRANSFER FUNCTIONS AND RESPONSES

3.1 Combination of components to physical system 52
3.2 Block diagram algebra and system reduction 53
3.2.1 Rules for block diagram reduction 54
3.2.2 Problems regarding moving take–off point ahead and behind summing points and vice-versa 62
3.3 Signal flow graph: (SFG) method 65
3.3.1 Introduction of signal flow graph and drawback of block diagram reduction technique 65
3.3.2 Construction to draw S.F.G from block diagram 65
3.3.3 Mason's gain formula 66
3.4 Time response analysis 69
3.4.1 Types of test signal 70
3.4.2 Time response analysis of first order system 71
3.4.3 Time response analysis of second order system 74
3.4.4 Transient response characteristics/specifications 84
3.5 Effect of feedback on steady state gain, bandwidth, error magnitude, system dynamics or response, stability, and parameter variation 87
3.5.1 Effect of feedback on system dynamics or response 87
3.5.2 Effect of feedback on bandwidth 87
3.5.3 Effect of feedback on steady gain 88
3.5.4 effect of feedback on stability 88
3.5.5 Effect of parameter variations [Robustness] 88
3.5.6 Effect of feedback on error magnitude (Accuracy) 89
3.5.7 Effect of disturbances on control system 89
3.6 Solved problems 91

Chapter 4 STABILITY

4.1 Introduction of stability and causes of instability 103
4.1.1 Introduction 103
4.1.2 Causes of instability 104
4.2 Characteristics equation, root location and stability 104
4.2.1 Characteristics equation 104
4.2.2 Root location and stability 104

Scanned with CamScanner

4.3	Routh–Hurwitz stability criterion	107
	4.3.1 Necessary but not sufficient condition for stability	107
	4.3.2 R–H criteria process	107
	4.3.3 Special cases of R–H stability criteria	110
4.4	Setting loop gain (K) using R–H criteria	118
4.5	Relative stability from complex plane axis shifting	121
	4.5.1 Relative and absolute stability	121
	4.5.2 Relative stability form complex plane shifting	122
4.6	Solved problems	125

Chapter 5 ROOT LOCUS TECHNIQUE

5.1	Introduction of root locus	137
	5.1.1 Angle criteria and magnitude criteria	138
5.2	Relationship between root loci and time response of system	140
	5.2.1 Location of roots of characteristics equation in S–plane	140
	5.2.2 Relationship	141
5.3	Rules for manual calculation and construction of root locus, analysis and design using root locus concept, stability analysis using R–H criteria	142
5.3.1	Effect of addition of poles and zeros on a root locus	151
5.4	Solved problems	153

Chapter 6 FREQUENCY RESPONSE TECHNIQUES

6.1	Frequency domain characterization of the system	172
6.2	Relationship between real and complex frequency response	172
	6.2.1 Frequency domain specification	173
6.3	Bode plot: magnitude and phase plot	173
	6.3.1 Bode plot of some standard form	174
	6.3.2 Resonant frequency and resonant peak for second order system	178
	6.3.3 Initial slope, initial magnitude and intersection with $\log_{10} \omega$ (i.e., with x–axis)	179
	6.3.4 General rule of construction of Bode-plot	180
	6.3.5 Minimum phase system and non-minimum phase systems	189
6.4	Effect of gain and time constant on Bode plot	189
	6.4.1 Effect of gain (K)	189
	6.4.2 Effect of time constant	189
6.5	Phase margin, gain margin, phase crossover frequency, gain crossover frequency and stability from Bode diagram	190
6.6	Polar plot and stability	191
	6.6.1 Polar plot	191

Scanned with CamScanner

6.6.2 Procedure to sketch polar plot 191
6.6.3 Polar plots of some standard type functions 192
6.6.4 Polar plot of type zero system 193
6.6.5 Polar plot of type I system 196
6.6.6 Polar plot of type II system 198
6.6.7 Effect of additional pole on type II system 198
6.6.8 Polar plot involving quadratic factor 200
6.6.9 Stability analysis from polar plot 201
6.7 Nyquist plot and stability 206
6.7.1 Mapping theory 207
6.7.2 Nyquist path, Nyquist contour and Nyquist plot 207
6.7.3 Stability criteria 208
6.7.4 Steps to sketch Nyquist plot 208
6.8 Solved problems 212

Chapter 7 PERFORMANCE SPECIFICATIONS AND COMPENSATION DESIGN

7.1. Time domain specification 224
7.1.1 Rise time, peak time, delay time, settling time and maximum overshoot 224
7.1.2 Static error coefficient 225
7.1.2.1 Steady state error (e_{ss}) 225
7.1.2.2 Static error coefficient 225
7.1.3 Steady state error for different types of system 227
7.2 Frequency domain specification 231
7.2.1 Gain margin and phase margin 231
7.3. Compensation and it's type 231
7.3.1 Compensation 231
7.3.2 Types of compensation 232
7.4 Design of compensation using Bode's plot 232
7.4.1 Phase lead compensation 232
7.4.2 Phase lag compensation (Pi controller) 241
7.5. Design of compensation using root locus 246
7.5.1 Design step for lead compensator 246
7.5.2 Design step for lag compensator 252
7.6 PID controllers 258
7.6.1 Controllers (Compensator) 258
7.6.1.1 Proportional controller (P–controller) 258
7.6.1.2 Derivative controller (D–controller) 259
7.6.1.3 Integral controller (I–controller) 260
7.6.1.4 PD controller (Lead compensator) 260
7.6.1.5 PI-controller (Lag compensator) 261
7.6.1.6 PID controller 262
7.7 Solved problems 263

Scanned with CamScanner

Chapter 8

STATE SPACE ANALYSIS

8.1	Definition of state-space	286
8.2	State space representation of electrical and mechanical systems	287
	8.2.1 State space representation of electrical system	287
	8.2.2 State space representation of differential equation	290
	8.2.3 State space representation of mechanical system	292
8.3	Conversion from state space to transfer function	293
	8.3.1 Stability analysis from state space	296
8.4	Conversion from transfer function (T.F.) to state space	298
	8.4.1 Transfer function containing poles only	298
	8.4.2 Transfer function containing both poles and zeros	300
	8.4.3 Controllability and observality	302
8.5	State transition matrix (STM)	303
8.6	Solved problems	306

Scanned with CamScanner

Chapter 1

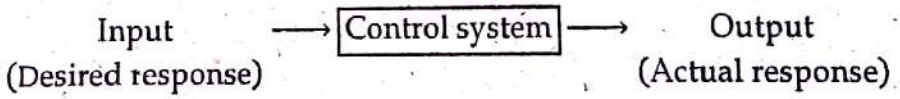
CONTROL SYSTEM BACKGROUND

1.1 History of control system and its importance 1
1.2 Control system : Characterisics and Basic Features 2
1.3 Types of control system and their comparison 3
1.3.1 Open loop control system (OLCS) 3
1.3.2 Closed loop control system (CLCS) 3
1.3.3 Comparison between open loop and closed loop system 4
1.3.4 Some example of control system 4

INTRODUCTION

Control system is that means by which any quantity of interest in a process/machine/mechanism/equipment etc. can be maintained or altered in accordance with desired manner. *For example;* Temperature control system (Air conditioner; AC), traffic light controller, washing machine, heating iron rod or immersion rod, etc.

Control system is any collection of components forming a system configuration that have cause and effect relationships among the system variable. It can also be defined as an interconnection of components that will provide a desired system response.



There are mainly two types of control system.

i) Open loop control system (OLCS)
ii) Closed loop control system (CLCS)

1.1 HISTORY OF CONTROL SYSTEM AND ITS IMPORTANCE

i) 1767

The first significant control device was James Watt's fly ball governor. This was invented to keep the speed of engine constant by regulating the supply of the steam to the engine.

ii) 1922

Minorsky developed automatic controller for ship steering. He also developed that behaviour of any control system can be described with the help of set of differential equations and so as its stability.

iii) 1932

Nyquist confirmed that stability of closed loop control system (CLCS) can be defined in easy way by observing response of open loop system with steady state sinusoidal inputs.

Scanned with CamScanner

iv) 1934,
Servo mechanism for positive control system

v) 1940,
Frequency response technique using Bode diagram. Here, we design a linear closed loop control system with related to specific performance.

vi) 1950,
Root locus method for design and stability. Frequency response technique and root locus technique for SISO (Single Input Single output).

vii) 1960,
State variable technique for MIMO (Multiple Input Multiple Output) was developed.

viii) 1960, to present
Complex control system

In Automatic control system, the existing values of a quantity or condition are measured and compare it with the desired value and the difference of these two values is used to initiate the action for reducing the difference.

There are many advantages of automatic control system.

a) Cost of energy or power reduces
b) Cost of processing materials in industries reduces
c) Quality of products improve
d) Productivity increases

1.2 CONTROL SYSTEM : CHARACTERISTICS AND BASIC FEATURES

Characteristics and features

The basis features of control system are shown below;

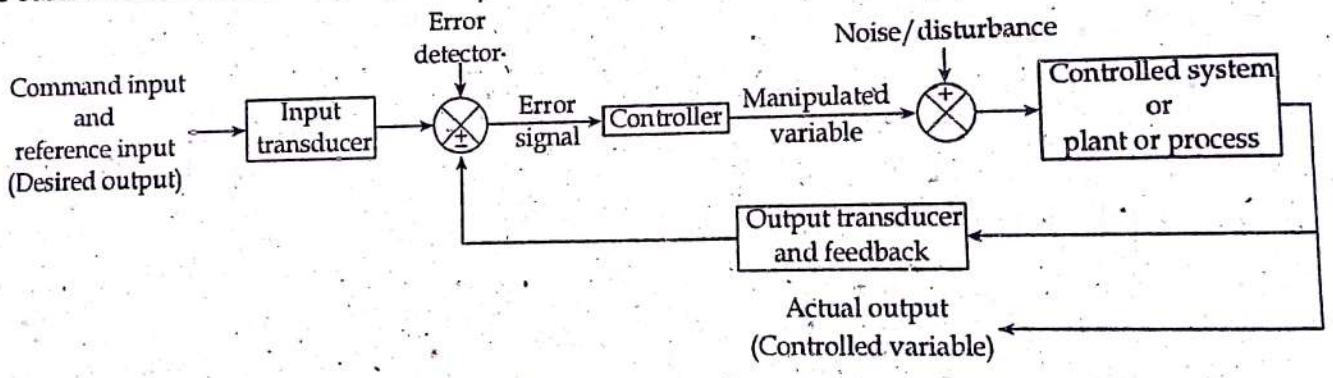


Figure: Components (block diagram) of a control system (CLCS)

i) Command Input
It is externally generated signal which generate reference input and independent of feedback path

ii) Reference Input
It is the desired parameter of control system to which our system is to be designed *i.e.*, desired value (or output) for the controlled variable.

iii) Input Transducer
Input transducer or input sensor converts the input signal. Transducer converts non-electrical signal into electrical signal.

iv) Error Detector
The errors detect gives out error signal. The error signal is defined as difference of desired output and actual output. The error should be as minimum as possible.

v) Controller or Control Element
It regulates the error signal form error detector. The controller may be OP-AMP, Relay or filter circuit.

Scanned with CamScanner

vi) Noise and Disturbance Signal

Unwanted signal such as: noise in amplifier, wind signal in RADAR etc. It affects the output of system.

vii) Controlled System/Plant/Process

It is the part of the system which does control

For example; Temperature control system

viii) Controlled Variable

The quantity which is measured and controlled.

ix) Manipulated Variable

It is that quantity which is varied by the controller so as to affect the value of controlled variable.

x) Feedback

The output transducer (or output sensor) senses the actual output of the system and passes it to the error detector in a regulated format. The feedback may be either positive or negative.

1.3 TYPES OF CONTROL SYSTEM AND THEIR COMPARISON

There are two main configurations of control system.

i) Open loop control system (OLCS)

ii) Closed loop control system (CLCS)

1.3.1 Open loop control system (OLCS)

- Non feedback control system.
- Control action is independent of the desired output.
- Output is not compared with the reference input
- The system can't work according to the actual output. In fact, it works according to reference signal.
- Open loop can be used in practice only if the relationship between the input and output is known and if there is no external or internal disturbances.
- *For example;* Traffic light controller, washing machine, immersion rod, etc.

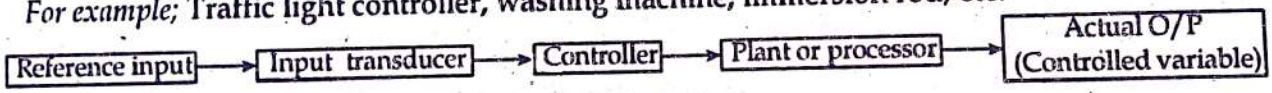


Figure: Block diagram of open loop control system

Advantages of OLCS

i) Simple

ii) Economical

iii) Les maintenance is required

iv) Proper calibration is not a problem

Disadvantages of OLCS

i) Open loop systems are less accurate

ii) These are not reliable

iii) These are slow

iv) Optimization is not possible

OLCS are "Time basis" control system

1.3.2 CLOSED LOOP CONTROL SYSTEM (CLCS)

- Also known as feedback control systems.
- Control action is dependent on the desired output.
- One or more feedback path.
- In this system the output is compared with the reference input and error signal is produced. The error is fed to the controller to reduce the error and desired output is obtained.
- Unwanted signal are highly reduced through feedback path and the system is fast.

Scanned with CamScanner

- CLCS are "Parameter basis" control system, other than time.
- As it is continuously maintaining the actual output at desired output, such systems are more reliable.
- *For example;* Air conditioner (AC), refrigerator, governor action in hydropower, etc.

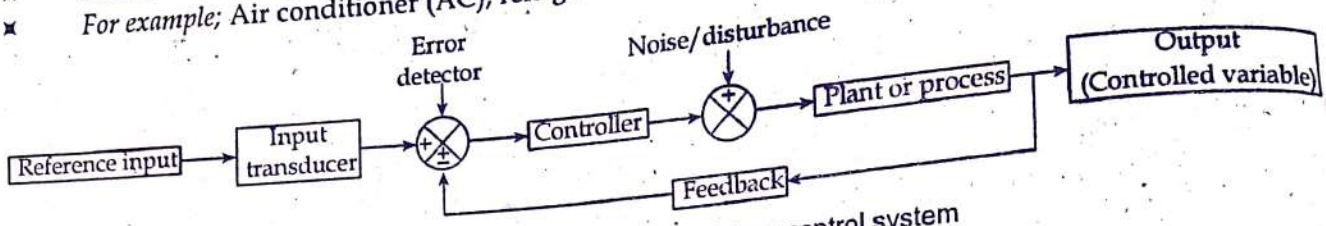


Figure: Block diagram of a Closed loop control system

Advantages of CLCS

i) These systems are more reliable.
ii) Closed loop system is faster
iii) A number of variables can be handled simultaneously
iv) Optimization is possible

Disadvantages of CLCS

i) Expensive
ii) Maintenance difficult
iii) Complex design
iv) Complicated installation

1.3.3 Comparison between open loop and closed loop system

S.N.	Open loop system	Closed loop system
1.	OLCS are simple and economical.	CLCS are complex and expensive.
2.	Less maintenance is required and easy for maintenance.	More maintenance is required and difficult for maintenance.
3.	These are not reliable.	These are reliable.
4.	If calibration is good, they perform accurately.	They are accurate because of feedback.
5.	Slower	Faster
6.	Optimization is not possible	Optimization is possible
7.	No feedback	With feedback

1.3.4 Some Example of Control System

Some open loop systems that we see in everyday life are discussed below:

i) Traffic Light Controller

For automatic control of traffic, the lamps of three different colours (red, green and yellow) are used. The time for each lamp is fixed. The operation of each lamp doesn't depend upon the density of the traffic but depends upon fixed time. Thus, we can say that the control system which operates on the time basis is open loop system.

ii) Immersion Rod

Immersion rod is another example of open loop system. The rod heats the water but how much heating is required is not sense by the rod because of no feedback to the rod.

iii) Washing Machine

Automatic washing machine is the example of the open loop control systems. In the machine, the operating time is set manually. After the completion of set time; the machine will stops, with the result

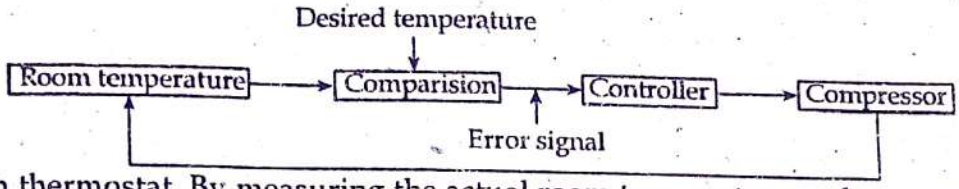
Scanned with CamScanner

we may or may not get the desired (output) amount of cleanliness of washed clothes because there is no feedback provided to the machine for desired output.

Some closed system that we see in everyday life is discussed below:

i) Automatic Temperature Control System [Air Conditioner; AC]

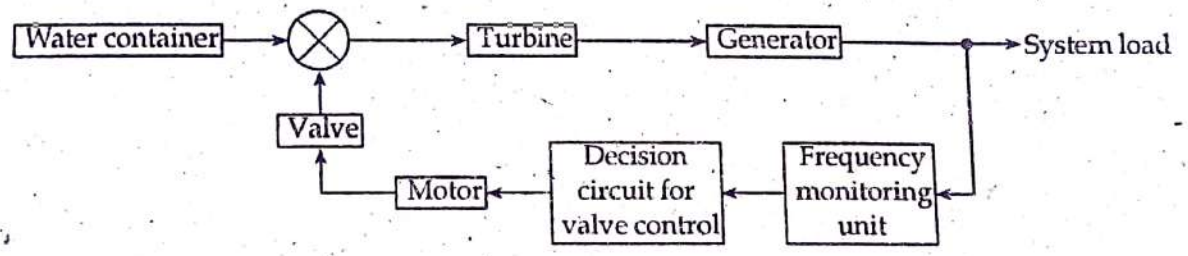
In a room we need to regulate the temperature and humidity for comfortable living. Air-conditioners



are provided with thermostat. By measuring the actual room temperature and compared it with desired temperature, an error signal is produced, the thermostat turns ON the compressor or OFF the compressor. The block diagram is shown below;

First of all, the temperature of a room is sensed by the temperature sensor which is then compared with the reference/desired temperature. The difference between actual and desired output is Error signal. The signal gives command signal to the controller so that the controller does necessary action to maintain desired condition on plant/process.

As it is continuously maintaining the actual output at desirable output, such system is more reliable, accurate and faster as well.



Thus, feedback mechanism plays pivotal role in control system which ultimately provides optimization.

ii) Governor action in hydropower

Governor Action in hydropower is the example of closed loop control system.

The governor is to receipt the frequency signal from the output of generator and it is compared with standard frequency signal. From these results, governor output signal is coming out to control the value of water at the entrance to the turbine.

To maintain the generator at a constant 50 Hz frequency, it is necessary to maintain the generator shaft at a constant rotational speed.

Scanned with CamScanner

1.4 SOLVED PROBLEMS

1. **What is feed forward compensation? Clearly show block diagram.**

Ans: In control system the disturbance has affected the output adversely. Such an output is measured and compared with the reference input to generate an error. This error is fed to the controller which successively operating the system to correct the input.

Thus, such systems in which the effect of the disturbance must show up in the error before the controller can take proper corrective action are called feedback system.

If the disturbance is measurable, the signal can be added to the controller output to modify the actuating signal. Thus, a corrective action is initiated without waiting for the effect of the disturbance to show up in the output *i.e.*, cumulatively in the error; thus, the undesirable effects of measureable disturbances by approximately compensating for them before they affect the output. This is much more advantageous as in normal feedback system the corrective action starts only after the output has been is affected. Such systems in which such corrective action is taken before disturbances affect the output are called feed forward system.

The block diagram for feed forward compensation is below;

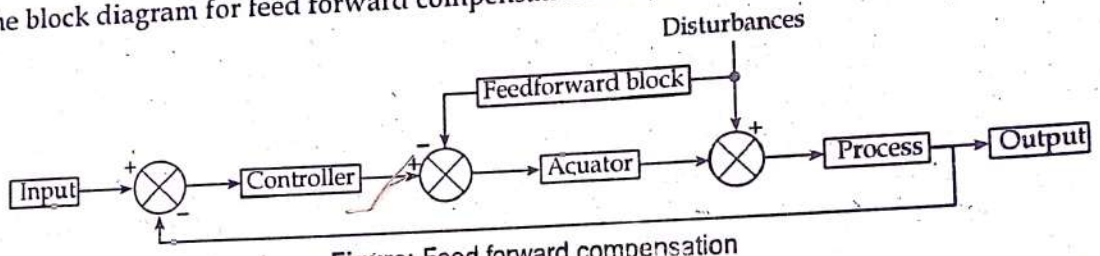


Figure: Feed forward compensation

2. **Draw the close loop control system configurations showing major components. Discuss the advantage and disadvantages of closed loop systems.** **[2065 Shrawan]**

Ans: See the definition part 1.2

3. **What is open loop and close loop control system? Draw the block diagram of closed loop control system and explain the role of each block.** **[2066 Bhadra]**

Ans: See the definition part 1.3 and 1.2

4. **Starting an example of a system that you see in everyone life, explain what do you understand by closed loop system and the importance of feedback in it?** **[2069 Chaitra]**

Ans: See the definition part 1.3.4

5. **What is control system? Draw the block diagram of a closed loop control system and briefly explain the function of each block. Mention also advantages of closed loop system over open loop system.** **[2071 Shrawan]**

Ans: See the definition part 1.2 and 1.3

6. **Construct a general block diagram of a control system showing the different blocks, variables and hence briefly point out their meanings.** **[2071 Chaitra]**

Ans: See the definition part 1.2

7. **What is control system? Mention its type, explain them with example, and block diagram showing all necessary blocks and signals.** **[2072 Kartik]**

Ans: See the definition part 1.2

8. **What kind of control system is governor system in hydropower station? Illustrate with necessary blocks and variables.** **[2073 Shrawan]**

Ans: See the definition part 1.3.4 (ii) [Examples of closed loop system]

9. **Define**
pronou

Ans: If a sys
(LTI) s
i.e., res
indivi
to an i

Mode
i)
ii)
iii)
iv)
v)
vi)

10. **Hov**
ana

Ans: See

Scanned with CamScanner

9. **Define linear time invariant system. Justify the statement "Modern complex systems are more pronounced with closed loop control system".** [2072 Chaitra]

Ans: If a system is both the linear and time invariant, then the system is called linear time invariant (LTI) system. A system is said to be linear where the superposition theorem can be implemented. *i.e.,* response produced by simultaneous application of two different inputs is the sum of two individual responses. A system is said to be time invariant if a time shift of the input signal leads to an identical time shift in the output signal.

Modern complex systems are more pronounced with closed loop control system because:

i) These systems are more reliable.
ii) Closed loop system is faster.
iii) Optimization is possible.
iv) A number of variables can be handled simultaneously.
v) These systems are more accurate.
vi) Closed loop system increases the bandwidth.

10. **How would a closed loop system differ from open loop one on its step response? Give analytical explanation.** [2074 Ashwin]

Ans: See the definition part 1.3.3

Scanned with CamScanner

Chapter 2

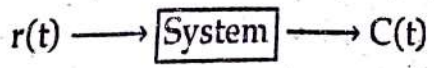
COMPONENT MODELING

2.1 Transfer function notation and differential equations 8
2.1.1 Characteristics equation of a system 9
2.1.2 Transfer function of closed loop control system 10
2.2 Modeling of mechanical components: mass, spring and damper 11
2.2.1 Resisting forces on a translational system 12
2.2.2 Resisting force on a rotational system 12
2.2.3 D' alembert's principle 13
2.2.4 Review of laplace transform 13
2.2.5 Concepts of mass-less body 18
2.3 Modeling of electrical components: inductance, resistance, capacitance, dc and ac motor, transducers and operational amplifier. 21
2.3.1 Resistance 21
2.3.2 Inductance 21
2.3.3 Capacitance 21
2.3.4 DC-motor 25
2.3.4.1 Armature control method for speed control of dc-motor (closed loop control system) 25
2.3.4.2 Field control method for speed control of dc-motor (open loop control system) 27
2.3.5 Operational amplifier (OP-Amp) 28
2.3.6 Transducers 31
2.4 Electrical circuit analogy 31
2.4.1 Force-voltage (F-V) analogy 31
2.4.2 Force-current (F-I) analogy 32
2.4.3 Torque voltage analogy: (T-V) analogy 36
2.4.4 Torque current analogy: (T-I) analog 36
2.5 Linearized approximation of non-linear characteristics 38

2.1 TRANSFER FUNCTION NOTATION AND DIFFERENTIAL EQUATIONS

Ratio of Laplace transform of output of a system to Laplace transform of input system with all initial condition as zero is termed as transfer function of a system.

Consider a system



Scanned with CamScanner

Any control system can be defined in terms of differential equations. Let us consider a system with input.

$$a_n\frac{d^n c(t)}{dt^n} + a_{n-1}\frac{d^{n-1}c(t)}{dt^{n-1}} + \ldots\ldots + a_1\frac{d^n c(t)}{dt} + a_0 = b_m\frac{d^m r(t)}{dt^m} + b_{m-1}\frac{d^{m-1}r(t)}{dt^{m-1}} + \ldots\ldots\ldots + b_1\frac{dr(t)}{dt} + b_0$$

Now taking Laplace transform on both sides;

$$[a_nS^n + a_{n-1}S^{n-1} + \ldots\ldots\ldots + a_1S + a_0]c(S) = [b_mS^m + b_{m-1}S^{m-1} + \ldots\ldots\ldots + b_1S + b_0]R(S)$$

and, $$\frac{C(S)}{R(S)} = \frac{b_mS^m + b_{m-1}S^{m-1} + \ldots\ldots\ldots + b_1S + b_0}{a_nS^n + a_{n-1}S^{n-1} + \ldots\ldots\ldots + a_1S + a_0}$$

We can define a transfer function as;

$$G(S) = \frac{C(S)}{R(S)} = \frac{L[c(t)]}{L[r(t)]}$$

$$\therefore \quad G(S) = \frac{C(S)}{R(S)} = \frac{b_mS^m + b_{m-1}S^{m-1} + \ldots\ldots\ldots + b_1S + b_0}{a_nS^n + a_{n-1}S^{n-1} + \ldots\ldots\ldots + a_1S + a_0}$$



Figure: SISO (single input single output) in S-domain

2.1.1 Characteristics Equation of a System

When the term in the denominator of transfer function is equalated with zero, it is termed as characteristics equation of a system. *i.e.,*

$$a_nS^n + a_{n-1}S^{n-1} + a_{n-2}S^{n-2} + \ldots\ldots\ldots + a_1S + a_0 = 0$$

Poles and Zeros of a System

Poles

The value of S for which $G(S) = \infty$

Zeros

The value of S for which $G(S) = 0$

For $G(S) = \dfrac{K(S + Z_1)(S + Z_2)\left(AS^2 + BS + C\right)}{(S + P_1)(S + P_2)\left(aS^2 + bS + C\right)}$

For poles;

$$S = -P_1, S = -P_2, S = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ are poles}$$

For zeros;

$$S = -Z_1, S = -Z_2, S = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \text{ are zeros}$$

and, K = Gain factor

For example;

EXAMPLE 2.1

Consider a system $G(S) = \dfrac{25(S + 1)(S + 2)}{(S + 3)(S + 4)^2(S + 5)}$; find gain factor, poles and zeros.

Solution:

Here,

Gain factor = 25

Zeros: $S = -1, S = -2$

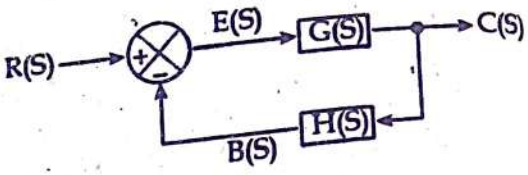
Scanned with CamScanner

Poles: $S = -3, S = -5$ and $S = -4$ multiple poles of order 2

and, S is the complex frequency.

2.1.2 Transfer Function of Closed Loop Control System

Consider a closed loop control system having forward Transfer function G (s) and backward transfer function H(s) as shown in the figure. The error detector gives the error signal E (s). Also,



Let, $R(S)$ = Laplace transform of input
$C(S)$ = Laplace transform of output
$E(S)$ = Laplace transform of Error
$G(S)$ = Forward transfer function
$H(S)$ = Backward/feedback transfer function

From the figure;

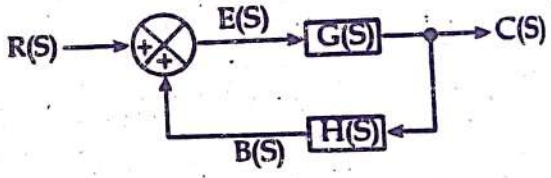
$$C(S) = E(S) \cdot G(S) \quad (1)$$
$$E(S) = R(S) - B(S) \quad (2)$$
$$B(S) = C(S) \cdot H(S) \quad (3)$$

From the above three equations; we have,

$$C(S) = E(S) \cdot G(S)$$
$$C(S) = [R(S) - B(S)] \cdot G(S)$$
$$C(S) = [R(S) - C(S) \cdot H(S)] \cdot G(S)$$
$$C(S) = R(S) \cdot G(S) - C(S) \cdot G(S) \cdot H(S)$$
$$C(S) + C(S) \cdot G(S) \cdot H(S) = R(S) \cdot G(S)$$
$$C(S) \cdot [1 + G(S) \cdot H(S)] = R(S) \cdot G(S)$$

$$\therefore \boxed{\frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)\,H(S)}}$$; for negative feedback

and, for positive feedback;



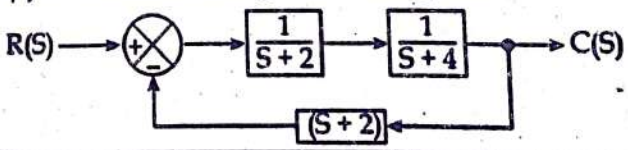
$$C(S) = E(S) \cdot G(S) \quad (1)$$
$$E(S) = R(S) + B(S) \quad (2)$$
$$B(S) = C(S) \cdot H(S) \quad (3)$$

Solving, these equations; we get transfer function of positive feedback closed loop system.

$$\therefore \boxed{\frac{C(S)}{R(S)} = \frac{G(S)}{1 - G(S)\,H(S)}}$$; for positive feedback

EXAMPLE 2.2

Find the transfer function (T.F.) of the following block diagram.



Solution:

For negative feedback;

$$\text{Transfer function (T.F.)} = \frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)\,H(S)}$$

Here,

$$G_1(S) = \frac{1}{(S+2)}$$

$$G_2(S) = \frac{1}{(S+4)}$$

Scanned with CamScanner

$$H(S) = (S+2)$$

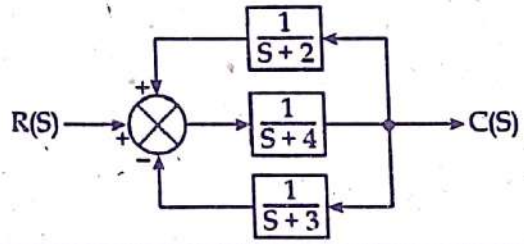
Then,

$$G(S) = G_1(S) \times G_2(S) = \frac{1}{(S+2)} \times \frac{1}{(S+4)}$$

so, $$\frac{C(S)}{R(S)} = \frac{\frac{1}{(S+2)(S+4)}}{1 + \frac{1}{(S+2)(S+4)} \times (S+2)} = \frac{1}{(S+2)(S+4)+(S+2)} = \frac{1}{(S+2)(S+4+1)} = \frac{1}{(S+2)(S+5)}$$

EXAMPLE 2.3

Find the transfer function (T.F.) of above circuit.



Solution:

For negative feedback;

$$\text{Forward (T.F.)} = G(S) = \frac{1}{(S+4)}$$

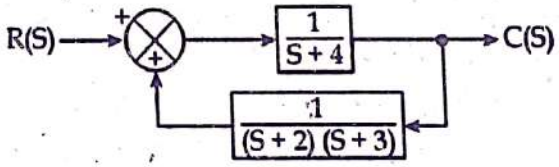
$$\text{Backward (T.F.)} = H(S) = H_1(S) - H_1(S)$$

where, $H_1(S) = \frac{1}{(S+2)}$

and, $H_2(S) = \frac{1}{(S+3)}$

so, $$H(S) = \frac{1}{(S+2)} - \frac{1}{(S+3)} = \frac{S+3-S-2}{(S+2)(S+3)} = \frac{1}{(S+2)(S+3)}$$

Now, above circuit can be reduced as;



Here, since we have done $H_1(S) - H_2(S) = H(S)$

so, for a positive feedback;

$$\text{T.F.} = \frac{C(S)}{R(S)} = \frac{G(S)}{1 - G(S)H(S)} = \frac{\frac{1}{(S+4)}}{1 - \frac{1}{(S+4)} \times \frac{1}{(S+2)(S+3)}} = \frac{(S+2)(S+3)}{(S+4)(S+2)(S+3)-1}$$

$$= \frac{S^2+5S+6}{S^3+9S^2+26S+23}$$

2.2 MODELING OF MECHANICAL COMPONENTS: MASS, SPRING AND DAMPER

Mechanical system is classified in to further two systems:

i) **Translational system**

Motion takes place in a straight line.

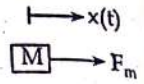
ii) **Rotational motion**

Motion of a body about a fixed axis.

Scanned with CamScanner

2.2.1 Resisting Forces on a Translational System

i) Inertia force



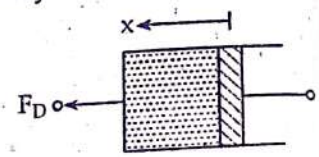
$$F_m = Ma = M\left(\frac{dV}{dt}\right) = M\left(\frac{d^2x}{dt^2}\right)$$

where, M is the mass.
V is the velocity.
x is the displacement.
and, t is the time.

$$\therefore \quad \boxed{F_m = M\ddot{x}}$$

ii) Damping force

For viscous friction we assume that the damping force is proportional to the velocity.

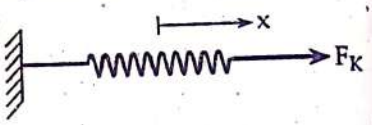


$$F_D \propto V$$
$$F_D = BV(t)$$
$$F_D = B \cdot \frac{d(x)}{dt}$$
$$\boxed{F_D = B \cdot \dot{x}}$$

where, B is the damping coefficient. [Unit: N/m/sec. or (NS/m)]

iii) Spring force

Spring stores the potential energy. The restoring force is proportional to the displacement.



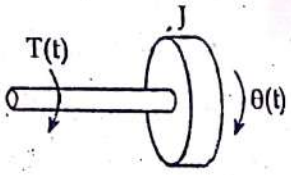
$$F_K \propto x \text{ (Hooke's law)}$$
$$F_K = Kx$$
$$\therefore \quad \boxed{F_K = Kx}$$

where, K is the spring constant or stiffness. [Unit: N/m].

2.2.2 Resisting Force on a Rotational System

i) Inertia torque

Inertia (J) is the property of an element that stores the kinetic energy of rotational motion.



$$T = J \cdot \alpha(t) = J \cdot \frac{d\omega(t)}{dt} = J \cdot \frac{d^2\theta}{dt^2}$$

$$\therefore \quad \boxed{T = J \cdot \ddot{\theta}}$$

where, J is the moment of inertia (MOI).
T is the inertia torque.
α is the angular acceleration.
ω is the angular velocity.
θ is the angular displacement.

Unit of inertia torque (T) is Nm.

ii) Damping torque

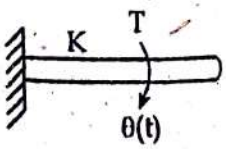
$$T = B \cdot \omega(t) = B \cdot \frac{d\theta(t)}{dt}$$

$$\therefore \quad \boxed{T = B\dot{\theta}}$$

iii) Spring torque

$$T = K \cdot \theta(t)$$

$$\therefore \quad \boxed{T = K\theta}$$



Scanned with CamScanner

The analogous form of translational and rotational motion is;

S.N.	Translational	Rotational
1.	Force (F)	Torque (T)
2.	Acceleration (a)	Angular acceleration (α)
3.	Velocity (V)	Angular velocity (w)
4.	Displacement (x)	Angular displacement (θ)
5.	Mass (M)	Moment of inertia (J)
6.	Damping coefficient (B)	Rotational Damping coefficient (B)
7.	Stiffness or spring constant (k)	Torsional stiffness (K)

2.2.3 D' Alembert's Principle

For translational system, "for anybody, algebraic sum of external forces and the forces resisting the body in any given direction is zero."

and, for rotational system, "for anybody, algebraic sum of external torques and torque resisting the body rotating about any axis is zero."

Consider a body, having mass, spring and damper as shown in figure (i).

Reference direction is along external applied force (F) downwards.

So, the resisting forces act upwardly [figure (ii)].

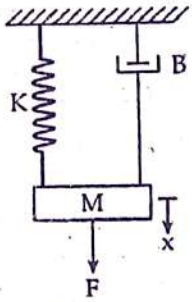


Figure: (i)

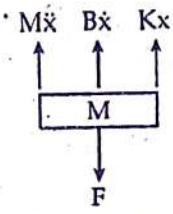


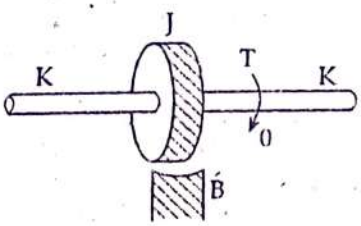
Figure: (ii)

so, form D' Alembert's principle;

$$F = (M\ddot{x} + B\dot{x} + Kx) = 0$$

or, $$\boxed{F = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx}$$; for translation system

and, for rotational motion;



so, from D' Alembert's principle;

$$T - (J\ddot{\theta} + B\dot{\theta} + K\theta) = 0$$

or, $$\boxed{T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta}$$; for rotational system

2.2.4 Review of Laplace Transform

If any function in time domain is f(t) then its Laplace form in s-domain is;

$$L[f(t)] = F(S) = \int_0^\infty e^{-st} f(t)dt$$

- Laplace transform of $\dot{x}(t)$

$$L[\dot{x}(t)] = S \cdot L[x(t)] = S \cdot x(S)$$

- Laplace transform of $\ddot{x}(t)$

$$L[\ddot{x}(t)] = S^2 \cdot L[x(t)] = S^2 \cdot x(S)$$

- Laplace transform of $\dddot{x}(t)$

$$L[\dddot{x}(t)] = S^3 \cdot L[x(t)] = S^3 \cdot x(S)$$

- Laplace transform of $\overset{n}{x}(t)$

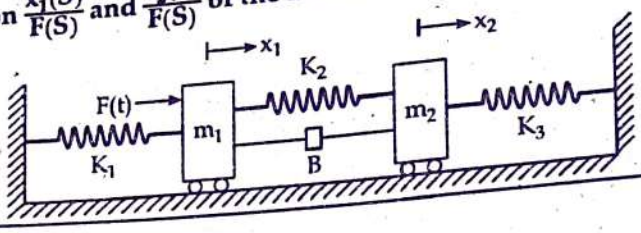
$$L\left[\overset{n}{x}(t)\right] = S^n \cdot L[x(t)] = S^n \cdot x(S)$$

Provided that all initial conditions [*i.e.*, $x(0)$] equals to zero.

Scanned with CamScanner

EXAMPLE 2.4

Obtain the transfer function $\frac{x_1(S)}{F(S)}$ and $\frac{x_2(S)}{F(S)}$ of the mechanical shown below.

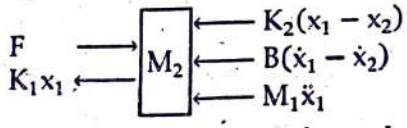


Solution:

No effect of friction on M_1 and M_2;

For mass M_1;

Drawing free body diagram (F.B.D.)



so, from D' Alembert's principle; we have,

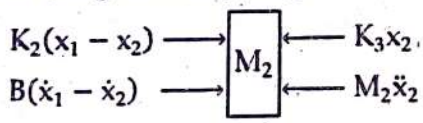
$$F = K_1x_1 + K_2(x_1 - x_2) + B(\dot{x}_1 - \dot{x}_2) + M_1\ddot{x}_1 \quad (1)$$

$$F = K_1x_1 + K_2x_1 + B\dot{x}_1 + M_1\ddot{x}_1 - K_2x_2 - B\dot{x}_2$$

Taking Laplace transform,

$$F(S) = [K_1 + K_2 + BS + S^2M_1]x_1(S) - [K_2 + BS]x_2(S) \quad (2)$$

Now, drawing free body diagram (F.B.D.) of M_2;



From D' Alembert's principle,

$$K_2(x_1 - x_2) + B(\dot{x}_1 - \dot{x}_2) = K_3x_2 + M_2\ddot{x}_2 \quad (3)$$

$$K_2x_1 - K_2x_2 + B\dot{x}_1 - B\dot{x}_2 = K_3x_2 + M_2\ddot{x}_2$$

$$K_2x_1 + B\dot{x}_1 = K_3x_2 + M_2\ddot{x}_2 + B\dot{x}_2 + K_2x_2$$

Taking Laplace transform;

$$[K_2 + BS]x_1(S) = [K_3 + S^2M_2 + BS + K_2]x_2(S)$$

$$\therefore \quad x_2(S) = \frac{(K_2 + BS)x_1(S)}{(K_2 + K_3 + BS + S^2M_2)} \quad (4)$$

Substituting the value of equation (4) in equation (2); we get,

$$F(S) = [K_1 + K_2 + BS + S^2M_1]x_1(S) - [K_2 + BS]x_2(S)$$

$$\text{or,} \quad F(S) = [K_1 + K_2 + BS + S^2M_1]x_1(S) - \frac{(K_2 + BS)(K_2 + BS)}{(K_2 + K_3 + BS + S^2M_2)}x_1(S)$$

$$\text{or,} \quad \frac{F(S)}{x_1(S)} = \frac{[(K_1 + K_2) + BS + S^2M_1][K_2 + K_3 + BS + S^2M_2] - (K_2 + BS)^2}{(K_2 + K_3 + BS + S^2M_2)}$$

$$\therefore \quad \boxed{\frac{x_1(S)}{F(S)} = \frac{(K_2 + K_3 + BS + S^2M_2)}{[(K_1 + K_2) + BS + S^2M_1][K_2 + K_3 + BS + S^2M_2] - (K_2 + BS)^2}} \quad (5)$$

so, again;

Scanned with CamScanner

$$\frac{x_2(S)}{F(S)} = \frac{x_1(S)}{F(S)} \times \frac{x_2(S)}{x_1(S)}$$

For equation **(4)** and **(5)**; we get,

$$\frac{x_2(S)}{F(S)} = \frac{x_1(S)}{F(S)} \times \frac{(K_2 + BS)}{\left(K_2 + K_3 + BS + S^2M_2\right)}$$

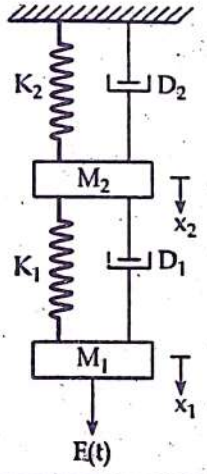
or, $$\frac{x_2(S)}{F(S)} = \frac{\left(K_2 + K_3 + BS + S^2M_2\right)}{\left[(K_1 + K_2) + BS + S^2M_1\right]\left[K_2 + K_3 + BS + S^2M_2\right] - (K_2 + BS)^2} \times \frac{(K_2 + BS)}{\left(K_2 + K_3 + BS + S^2M_2\right)}$$

$$\therefore \quad \boxed{\frac{x_2(S)}{F(S)} = \frac{(K_2 + BS)}{\left[K_1 + K_2 + BS + S^2M_1\right]\left[K_2 + K_3 + BS + S^2M_2\right] - (K_2 + BS)^2}} \qquad (6)$$

Equation (5) and (6) are the required solutions.

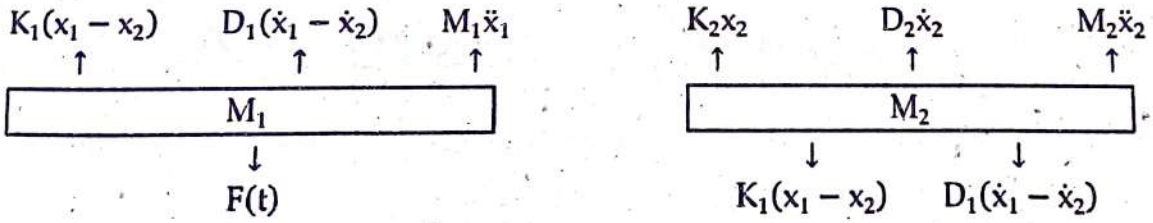
EXAMPLE 2.5

Draw the free diagram, write the differential equations and find out $\frac{x_2(S)}{F(S)}$ and $\frac{x_1(S)}{F(S)}$ of the system below.



Solution:

Force body diagram of M_1 and M_2 are;



From De-Alembert's principle;

For M_1;

$$F(t) = K_1(x_1 - x_2) + D_1(\dot{x}_1 - \dot{x}_2) + M_1\ddot{x}_1$$

$$F(t) = K_1x_1 + D_1\dot{x}_1 + M_1\ddot{x}_1 - K_1x_2 - D_1\dot{x}_2 \qquad (1)$$

Taking L.T. of equation (1); we get,

$$F(S) = \left[K_1 + D_1S + S^2M_1\right]x_1(S) - \left[K_1 + D_1S\right]x_2(S) \qquad (2)$$

For M_2; applying D' Alembert's principle; we have,

$$K_2x_2 + D_2\dot{x}_2 + M_2\ddot{x}_2 = K_1(x_1 - x_2) + D_1(\dot{x}_1 - \dot{x}_2)$$

or, $$K_2x_2 + D_2\dot{x}_2 + M_2\ddot{x}_2 = K_1x_1 - K_1x_2 + D_1\dot{x}_1 - D_1\dot{x}_2$$

or, $$K_2x_2 + D_2\dot{x}_2 + M_2\ddot{x}_2 + K_1x_2 + D_1\dot{x}_2 = K_1x_1 + D_1\dot{x}_1 \qquad (3)$$

Scanned with CamScanner

Taking L.T. of equation (3); we have,

$$[K_1 + D_1S]x_1(S) = [K_2 + D_2S + S^2M_2 + K_1 + D_1S]x_2(S) \quad (4)$$

or, $$[K_1 + D_1S]x_1(S) - [K_2 + K_1 + D_1S + D_2S + S^2M_2]x_2(S) = 0$$

Now, let us do this by the help of matrix method;

Writing equation (2) and (4) in the matrix form;

$$\begin{bmatrix}(K_1 + D_1S + S^2M_1) & -(K_1 + D_1S)\\ (K_1 + D_1S) & -(K_2 + K_1 + D_1S + D_2S + S^2M_2)\end{bmatrix}\begin{bmatrix}x_1(S)\\ x_2(S)\end{bmatrix} = \begin{bmatrix}F(S)\\ 0\end{bmatrix}$$

Here,

$$\Delta = \{-(K_1 + D_1S + S^2M_1)(K_2 + K_1 + D_1S + D_2S + S^2M_2) + (K_1 + D_1S)\}$$

$$\Delta_1 = \begin{vmatrix}F(S) & -(K_1 + D_1S)\\ 0 & -(K_2 + K_1 + D_1S + D_2S + S^2M_2)\end{vmatrix} = -(K_2 + K_1 + D_1S + D_2S + S^2M_2)F(S)$$

and, $$\Delta_2 = \begin{vmatrix}(K_1 + D_1S + S^2M_1) & F(S)\\ (K_1 + D_1S) & 0\end{vmatrix} = -F(S)(K_1 + D_1S)$$

Therefore, applying Cramer's rule; we have,

$$x_1(S) = \frac{\Delta_1}{\Delta}$$

and, $$x_2(S) = \frac{\Delta_2}{\Delta}$$

so, $$x_1(S) = \frac{\left(K_1 + K_2 + D_1S + D_2S + S^2M_2\right)F(S)}{(K_1 + D_1S)^2 - \left(K_1 + D_1S + S^2M_1\right)\left(K_1 + K_2 + D_1S + D_2S + S^2M_2\right)}$$

$$\therefore \quad \boxed{\frac{x_1(S)}{F(S)} = \frac{\left(K_1 + K_2 + (D_1 + D_2)S + S^2M_2\right)}{\left(K_1 + D_1S + S^2M_1\right)\left(K_1 + K_2 + D_1S + D_2S + S^2M_2\right) - (K_1 + D_1S)^2}} \quad (5)$$

and, $$x_2(S) = \frac{-F(S)(K_1 + D_1S)}{(K_1 + D_1S)^2 - \left(K_1 + D_1S + S^2M_1\right)\left(K_1 + K_2 + D_1S + D_2S + S^2M_2\right)}$$

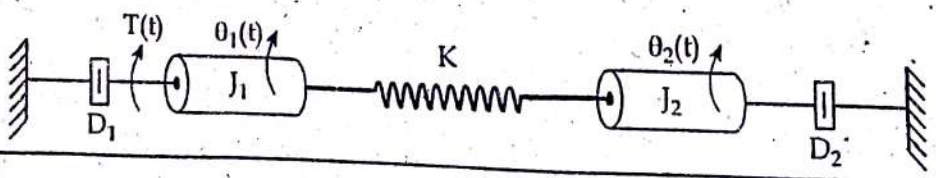
$$\therefore \quad \boxed{\frac{x_2(S)}{F(S)} = \frac{(K_1 + D_1S)}{\left(K_1 + D_1S + S^2M_1\right)\left(K_1 + K_2 + D_1S + D_2S + S^2M_2\right) - (K_1 + D_1S)^2}} \quad (6)$$

Hence, equation (5) and (6) are the required expressions.

NOTE
We can do it with the help of substitution method as well.

EXAMPLE 2.6

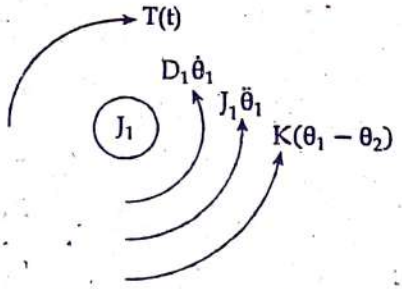
Find out $\frac{\theta_2(S)}{T(S)}$ for the rotational system.



Solution:

Free body diagram of body having MOI J_1 is;

Scanned with CamScanner



Here, reference direction is along torque T(t) which is clockwise. So, the restoring torques is in anticlockwise fashion.

From De' Alembert's principle; we have,

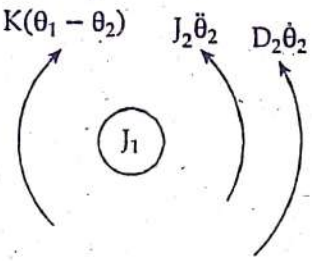
$$T(t) = D_1\dot{\theta}_1 + J_1\ddot{\theta}_1 + K(\theta_1 - \theta_2)$$

or, $$T(t) = D_1\dot{\theta}_1 + J_1\ddot{\theta}_1 + K\theta_1 - K\theta_2 \quad (1)$$

Taking, Laplace transform;

$$T(S) = [K + D_1S + S^2J_1]\theta_1(S) - K\theta_2(S) \quad (2)$$

Now, free body diagram of body having MOI J_2 is;



From De' Alembert's principle; we have,

$$K(\theta_1 - \theta_2) = J_2\ddot{\theta}_2 + D_2\dot{\theta}_2$$

or, $$K\theta_1 - K\theta_2 = J_2\ddot{\theta}_2 + D_2\dot{\theta}_2$$

or, $$K\theta_1 = J_2\ddot{\theta}_2 + D_2\dot{\theta}_2 + K\theta_2 \quad (3)$$

Taking L.T. of equation (3); we get,

$$K\theta_1(S) = (K + D_2S + S^2J_2)\theta_2(S)$$

or, $$K\theta_1(S) - (K + D_2S + S^2J_2)\theta_2(S) = 0 \quad (4)$$

Now, arranging equation (2) and (4) in matrix form;

$$\begin{bmatrix} (K + D_1S + S^2J_1) & -K \\ K & -(K + D_1S + S^2J_2) \end{bmatrix} \begin{bmatrix} \theta_1(S) \\ \theta_2(S) \end{bmatrix} = \begin{bmatrix} T(S) \\ 0 \end{bmatrix}$$

Now,

$$\Delta = -(K + D_1S + S^2J_1)(K + D_2S + S^2J_2) + K^2$$

$$\Delta_1 = \begin{vmatrix} T(S) & -K \\ K & -(K + D_1S + S^2J_2) \end{vmatrix} = -(K + D_1S + S^2J_2)T(S)$$

and, $$\Delta_2 = \begin{vmatrix} K + D_1S + S^2J_2 & T(S) \\ K & 0 \end{vmatrix} = -KT(S)$$

From Cramer's rule; we have,

$$\theta_1(S) = \frac{\Delta_1}{\Delta} = \frac{-(K + D_1S + S^2J_2)T(S)}{-(K + D_1S + S^2J_1)(K + D_2S + S^2J_2) + K^2}$$

$$\therefore \quad \boxed{\frac{\theta_1(S)}{T(S)} = \frac{(K + D_1S + S^2J_2)}{(K + D_1S + S^2J_1)(K + D_2S + S^2J_2) + K^2}} \quad (5)$$

Scanned with CamScanner

Again,

$$\theta_2(S) = \frac{\Delta_2}{\Delta} = \frac{-KT(S)}{-\left(K + D_1S + S^2J_1\right)\left(K + D_2S + S^2J_2\right) + K^2}$$

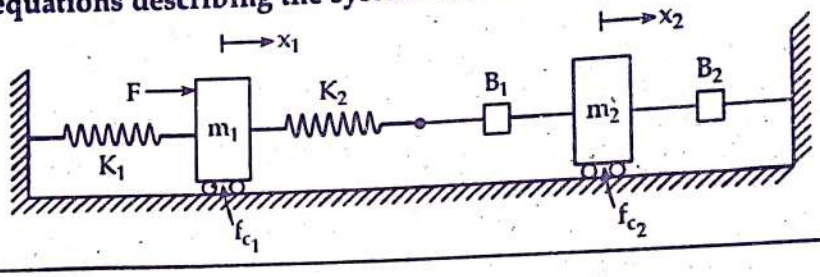
$$\therefore \quad \boxed{\frac{\theta_2(S)}{T(S)} = \frac{K}{\left(K + D_1S + S^2J_1\right)\left(K + D_2S + S^2J_2\right) + K^2}} \qquad (6)$$

Equations (5) and (6) are the required expressions.

2.2.5 Concepts of Mass–less Body

EXAMPLE 2.7

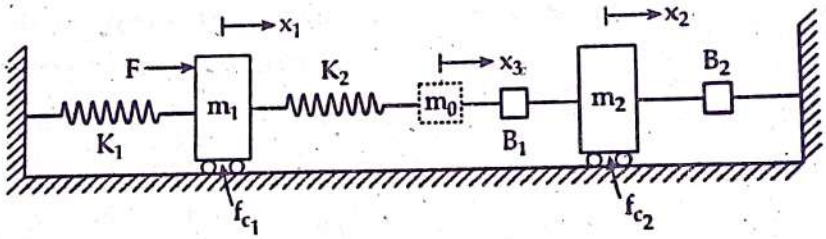
Write mathematical equations describing the system and also find the transfer functions.



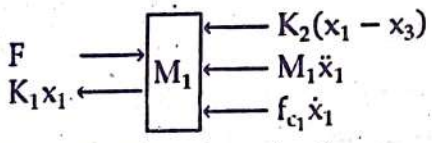
Solution:

Here, two Resting forces (K_2 and B_1) are acting serially. So we have to consider a Mass less body which have mass Mo and displacement x_3 (say). The body M_0 is in between K_2 and B_1.

So, the block becomes;



Now, F.B.D. of M_1;



From the principle;

$$F = K_1x_1 + K_2(x_1 - x_3) + M_1\ddot{x}_1 + f_{c_1}\dot{x}_1$$

or, $$F = K_1x_1 + K_2x_1 + f_{c_1}\dot{x}_1 + M_1\ddot{x}_1 - K_2x_3 \qquad (1)$$

Taking L.T. of equation (1); we have,

$$F(S) = \left[K_1 + K_2 + Sfc_1 + S^2M_1\right]x_1(S) - K_2x_3(S) \qquad (2)$$

Now, F.B.D. of M_0;

$$K_2(x_1 - x_3) \longrightarrow \boxed{M_0} \longleftarrow B_1(\dot{x}_3 - \dot{x}_2)$$

and, $M_0 = 0$

From the principle; we have,

$$K_2(x_1 - x_3) = B_1(\dot{x}_3 - \dot{x}_2)$$

or, $$K_2x_1 - K_2x_3 = B_1\dot{x}_3 - B_1\dot{x}_2$$

or, $$K_2x_1 + B_1\dot{x}_2 = K_2x_3 + B_1\dot{x}_3 \qquad (3)$$

Taking L.T. of equation (3); we have,

$$\left[K_2 + SB_1\right]x_3(S) = K_2x_1(S) + B_1Sx_2(S)$$

Scanned with CamScanner

$$\therefore \quad x_3(S) = \frac{K_2x_1(S) + B_1Sx_2(S)}{(K_2 + SB_1)} \tag{4}$$

Again, F.B.D. of M_2;

$B_1(\dot{x}_3 - \dot{x}_2) \longrightarrow$ [M_2] $\longleftarrow B_2\dot{x}_2$, $\longleftarrow f_{c_2}\dot{x}_2$, $\longleftarrow M_2\ddot{x}_2$

From the principle; we have,

$$B_1(\dot{x}_3 - \dot{x}_2) = B_2\dot{x}_2 + f_{c_2}\dot{x}_2 + M_2\ddot{x}_2$$

$$B_1\dot{x}_3 = B_1\dot{x}_2 + B_2\dot{x}_2 + f_{c_2}\dot{x}_2 + M_2\ddot{x}_2 \tag{5}$$

Taking L.T. of equation (5); we have,

$$B_1S\,x_3(S) = [B_1S + B_2S + f_{c_2}S + S^2M_2]x_2(S)$$

$$\therefore \quad x_3(S) = \frac{(B_1 + B_2 + f_{c_2} + SM_2)x_2(S)}{B_1} \tag{6}$$

Substituting the value of equation (6) in the equation (4); we get,

$$\frac{(B_1 + B_2 + f_{c_2} + SM_2)x_2(S)}{B_1} = \frac{K_2x_1(S) + B_1S\,x_2(S)}{(K_2 + SB_1)}$$

$$\text{or,} \quad (B_1 + B_2 + f_{c_2} + SM_2)(K_2 + SB_1)x_2(S) = K_2B_1x_1(S) + B_1^2Sx_2(S)$$

$$\text{or,} \quad x_1(S) = \frac{\left[(B_1 + B_2 + f_{c_2} + SM_2)(K_2 + SB_1) - B_1^2S\right]x_2(S)}{K_2B_1} \tag{7}$$

Substituting the value of equation (6) and (7) into equation (2); we get,

$$F(S) = [K_1 + K_2 + Sf_{c_1} + S^2M_1]x_1(S) - K_2x_3(S)$$

$$\text{or,} \quad F(S) = [K_1 + K_2 + Sf_{c_1} + S^2M_1]\frac{\left[(B_1 + B_2 + f_{c_2} + SM_2)(K_2 + SB_1) - B_1^2S\right]}{K_2B_1}x_2(S) - K_2\frac{(B_1 + B_2 + f_{c_2} + SM_2)}{B_1}x_2(S)$$

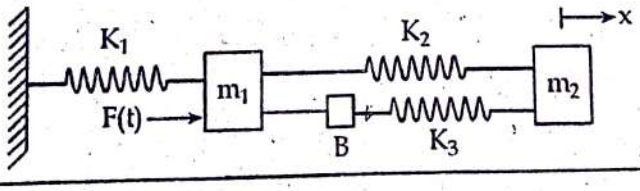
$$\text{or,} \quad \frac{F(S)}{x_2(S)} = \frac{\left[K_1 + K_2 + Sf_{c_1} + S^2M_1\right]\left[(B_1 + B_2 + f_{c_2} + SM_2)(K_2 + SB_1) - B_1^2S\right] - K_2^2(B_1 + B_2 + f_{c_2} + SM_2)}{K_2B_1}$$

$$\therefore \quad \boxed{\frac{x_2(S)}{F(S)} = \frac{K_2B_1}{\left[K_1 + K_2 + Sf_{c_1} + S^2M_1\right]\left[(B_1 + B_2 + f_{c_2} + SM_2)(K_2 + SB_1) - B_1^2S\right] - K_2^2(B_1 + B_2 + f_{c_2} + SM_2)}} \tag{8}$$

Equation (8) is the required transfer function.

EXAMPLE 2.8

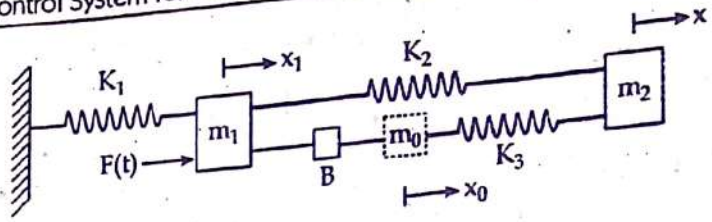
Find transfer function for the following mechanical system considering displacement of mass M_2 as output of the system.



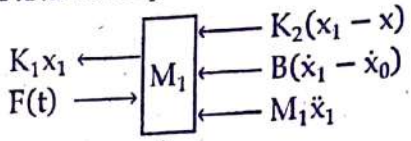
Solution:

Here, B and K_3 act serially inside two blocks M_1 and M_2; so, we have to consider the effect of mass-less mass. The displacement of mass-less body is x_0 and that of M_2 is x_1 (say).

Scanned with CamScanner



Now, F.B.D. of M_1;



Then,

$$F(t) = K_1x_1 + K_2(x_1 - x) + B(\dot{x}_1 - \dot{x}_0) \quad (1)$$

or, $$F(t) = K_1x_1 + K_2x_1 + B\dot{x}_1 - K_2x - B\dot{x}_0 \quad (2)$$

Taking L.T. of equation (1); we have,

$$F(S) = [K_1 + K_2 + SB]x_1S - K_2x(S) - BSx_0(S)$$

F.B.D. of mass-less mass ($M_0 = 0$) is;

$$B(\dot{x}_1 - \dot{x}_0) \longrightarrow \boxed{M_0} \longleftarrow K_3(x_0 - x)$$

so, $$B(\dot{x}_1 - \dot{x}_0) = K_3(x_0 - x)$$

or, $$B\dot{x}_1 - B\dot{x}_0 = K_3x_0 - K_3x \quad (3)$$

or, $$B\dot{x}_1 + K_3x = K_3x_0 + B\dot{x}_0$$

Taking L.T. of equation (3); we get,

$$[K_3 + SB]x_0(S) = BSx_1(S) + K_3x(S)$$

or, $$x_0(S) = \frac{BSx_1(S) + K_3x(S)}{(K_3 + SB)} \quad (4)$$

F.B.D. of mass M_2 is;

$$K_2(x_1 - x) \longrightarrow \boxed{M_2} \longleftarrow M_2\ddot{x}$$
$$K_3(x_0 - x) \longrightarrow$$

so, $$K_2(x_1 - x) + K_3(x_0 - x) = M_2\ddot{x}$$

or, $$K_2x_1 + K_3x_0 = K_3x + K_2x + M_2\ddot{x} \quad (5)$$

Taking Laplace transform of equation (5); we get,

$$[K_3 + K_2 + S^2M_2]x(S) = K_2x_1(S) + K_3x_0(S) \quad (6)$$

From the equation (4) and (6); we have,

Substituting the value of $x_0(S)$ in the equation (6); we have,

$$(K_3 + K_2 + S^2M_2)x(S) = K_2x_1(S) + K_3\left[\frac{BSx_1(S) + K_3x(S)}{(K_3 + SB)}\right]$$

or, $$(K_3 + K_2 + S^2M_2)x(S) = \frac{K_2(K_3 + SB) + K_3[BSx_1(S) + K_3x(S)]}{(K_3 + SB)}$$

or, $$(K_3 + K_2 + S^2M_2)(K_3 + SB)x(S) = [K_2(K_3 + SB) + K_3BS]x_1(S) + K_3^2x(S)$$

$$\therefore \quad x_1(S) = \frac{\left[(K_2 + K_3 + S^2M_2)(K_3 + SB) - K_3^2\right]x(S)}{(K_2K_3 + K_2BS + K_3BS)} \quad (7)$$

Now, substituting the value of (4) in (2); we get,

$$F(S) = (K_1 + K_2 + SB)x_1(S) - K_2x(s) - BSx_0(S)$$

$$= (K_1 + K_2 + SB)x_1(S) - K_2x(s) - BS\left[\frac{BSx_1(S) + K_3x(S)}{(K_3 + SB)}\right]$$

Scanned with CamScanner

$$= (K_1 + K_2 + SB)x_1(S) - \frac{B^2S^2x_1(S)}{(K_3 + SB)} - K_2x(s) - \frac{K_3BSx(S)}{(K_3 + SB)}$$

$$= \left[\frac{(K_3 + SB)(K_1 + K_2 + SB) - B^2S^2}{(K_3 + SB)}\right]x_1(S) - \left[\frac{K_2(K_3 + BS) - K_3BS}{(K_3 + BS)}\right] \quad (8)$$

Substituting the value of (7) in (8); we get,

$$\frac{F(S)}{x(S)} = \frac{(K_3 + SB)(K_1 + K_2 + SB) - B^2S^2}{(K_3 + SB)} \times \frac{(K_3 + SB)\left(K_2 + K_3 + S^2M_2\right) - K_3^2}{K_2K_3 + K_2SB + K_3SB} - \left[\frac{K_2(K_3 + SB) - K_3SB}{K_3 + SB}\right]$$

$$\left[(K_3 + SB)(K_1 + K_2 + SB) - S^2B^2\right]\left[(K_3 + SB)\left(K_2 + K_3 + S^2M_2\right) - K_3^2\right] - (K_2K_3 + K_2SB + K_3SB)$$

or, $$\frac{F(S)}{x(S)} = \frac{(K_2K_3 + K_2SB + K_3SB)}{(K_3 + SB)(K_2K_3 + K_2SB + K_3SB)}$$

$$\therefore \quad \frac{x(S)}{F(S)} = \frac{(K_3 + SB)(K_2K_3 + K_2SB + K_3SB)}{\left[(K_3 + SB)(K_1 + K_2 + SB) - S^2B^2\right]\left[(K_3 + SB)\left(K_2 + K_3 + S^2M_2\right) - K_3^2\right] - \left[(K_2K_3 + K_2SB)^2 - K_3^2S^2B^2\right]} \quad (9)$$

Equation (9) is the required expression for $\frac{x(S)}{F(S)}$.

2.3 MODELING OF ELECTRICAL COMPONENTS: INDUCTANCE, RESISTANCE, CAPACITANCE, DC AND AC MOTOR, TRANSDUCERS AND OPERATIONAL AMPLIFIER.

2.3.1 Resistance

We know that;

$$V = IR$$

where, V is the voltage (V).

I is the current (Amp).

R is the resistance (Ohm).

Taking Laplace transform;

$$V(S) = I(S) \times R$$

$$\therefore \quad \frac{V(S)}{I(S)} = R$$

i.e., $\boxed{R(S) = R}$

Laplace transform of resistance is R.

2.3.2 Inductance

We know that;

$$V = L\frac{dI}{dt}$$

where, L is the inductance (H).

Taking Laplace transform; we have,

$$V(S) = L \cdot SI(S)$$

or, $$\frac{V(S)}{I(S)} = LS$$

$$\therefore \quad \boxed{L(S) = SL}$$

Laplace transform of inductance is LS.

2.3.3 Capacitance

We know that;

Scanned with CamScanner

$$I = C\frac{dV}{dt}$$

where, C is the capacitance (F).

Taking Laplace transform; we have,

$$I(S) = C \cdot SV(S)$$

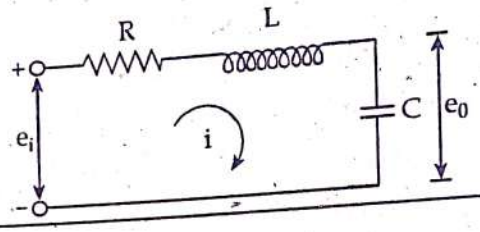
or, $$\frac{1}{CS} = \frac{V(S)}{I(S)}$$

$$\therefore \quad \boxed{C(S) = \frac{1}{CS}}$$

Laplace transform of capacitance is $\left(\frac{1}{CS}\right)$.

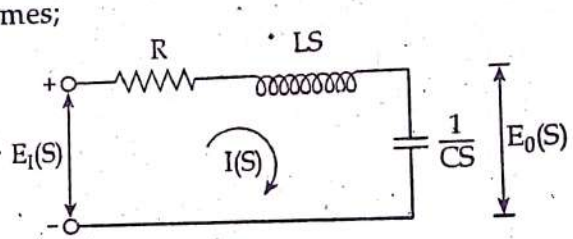
EXAMPLE 2.9

For the given circuit, find $\frac{E_0(S)}{E_i(S)}$.



Solution:

Above circuit in s-domain becomes;



Here,

Laplace transform of R = R

Laplace transform of L = LS

Laplace transform of $C = \frac{1}{CS}$

Laplace transform of I = I(S)

Laplace transform of $e_i = E_I(S)$

Laplace transform of $e_0 = E_0(S)$

Now, applying KVL to above circuit; we have,

$$E_I(S) = \left[R + LS + \frac{1}{CS}\right] \times I(S)$$

and, output voltage is across the capacitor; so,

$$E_0(S) = \frac{1}{CS} \times I(S)$$

Dividing equation **(2)** by **(1)**; we get,

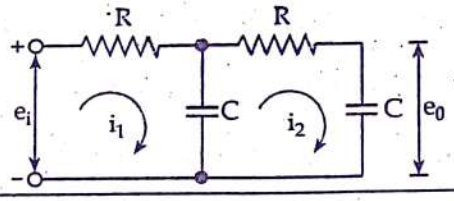
$$\frac{E_0(S)}{E_I(S)} = \frac{\frac{1}{CS} \times I(S)}{\left(R + LS + \frac{1}{CS}\right) \times I(S)}$$

$$\therefore \quad \boxed{\frac{E_0(S)}{E_I(S)} = \frac{1}{1 + RCS + LCS^2}}$$

Scanned with CamScanner

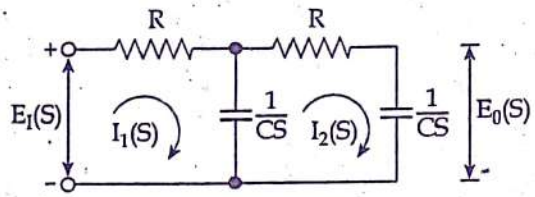
EXAMPLE 2.10

For the given circuit, find transfer function of a system.



Solution:

The circuit in S-domain becomes;



Applying KVL in loop (1);

$$E_I(S) = R \cdot I_1(S) + \frac{1}{CS}[I_1(S) - I_2(S)] = \left(R + \frac{1}{CS}\right) I_1(S) - \frac{I_2(S)}{CS} = \left(\frac{RCS+1}{CS}\right) I_1(S) - \frac{I_2(S)}{C(S)} \quad (1)$$

Again, applying K.V.L. in loop (2);

$$\left(R + \frac{1}{CS}\right) I_2(S) + \frac{1}{CS}[I_2(S) - I_1(S)] = 0$$

or, $$\left(\frac{RCS+1}{CS}\right) I_2(S) + \frac{1}{CS}[I_2(S) - I_1(S)] = 0$$

or, $$(RCS+1)I_2(S) + I_2(S) - I_1(S) = 0$$

or, $$I_2(S)(RCS+2) - I_1(S) = 0$$

$\therefore$ $$I_i(S) = (RCS+2)I_2(S) \quad (2)$$

Again,

$$\text{Output voltage } [E_0(S)] = \frac{1}{CS} \times I_2(S) \quad (3)$$

Substituting the value of (2) in the equation (1); we have,

$$E_I(S) = \left(\frac{RCS+1}{CS}\right) \times (RCS+2)I_2(S) - \frac{I_2(S)}{CS}$$

or, $$E_I(S) = [(RCS+1)(RCS+2) - 1] \cdot \frac{I_2(S)}{CS} \quad (4)$$

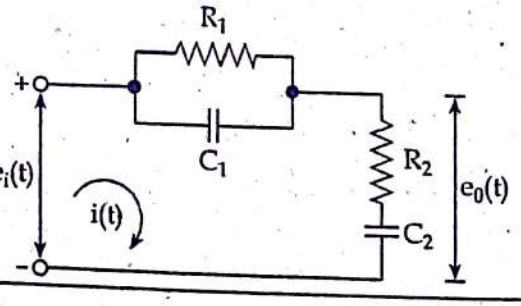
Substituting the value of equation (3) in (4); we have,

$$E_I(S) = (R^2C^2S^2 + 3RCS + 1) \cdot E_0(S)$$

$\therefore$ $$\boxed{\frac{E_0(S)}{E_I(S)} = \frac{1}{1 + 3RCS + R^2C^2S^2}}$$

EXAMPLE 2.11

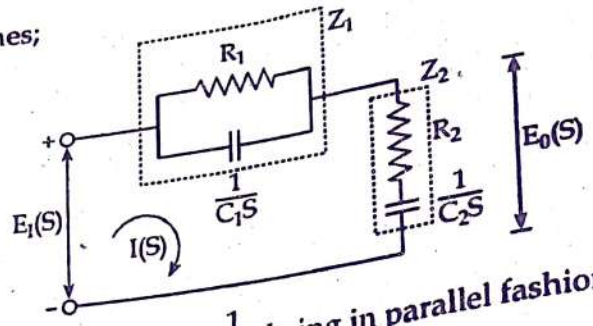
For the circuit, find transfer function of a system.



Scanned with CamScanner

Solution:

The circuit in S-domain becomes;

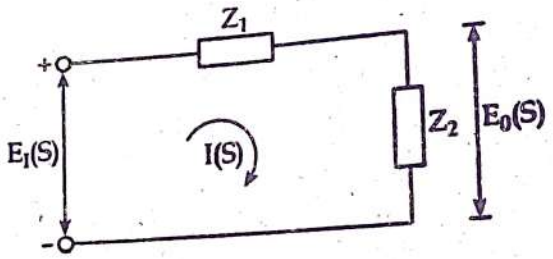


Let us consider a block Z_1 such that $Z_1 = R_1 \parallel \frac{1}{C_1S}$ being in parallel fashion.

$$\therefore \quad Z_1 = \frac{R_1 \times \frac{1}{C_1S}}{R_1 + \frac{1}{C_1S}} = \frac{R_1}{R_1C_1S + 1}$$

and, $Z_2 = \left[R_2 + \frac{1}{C_2S}\right] = \left[\frac{R_2C_2S + 1}{C_2S}\right]$; being in serial connection

The circuit becomes;



so, applying KVL;

$$E_I(S) = (Z_1 + Z_2) \cdot I(S)$$

$$\text{or,} \quad E_I(S) = \left(\frac{R_1}{R_1C_1S + 1} + \frac{R_2C_2S + 1}{C_2S}\right) \cdot I(S)$$

$$\text{or,} \quad E_I(S) = \left[\frac{R_1C_2S + R_1R_2C_1C_2S^2 + R_1C_1S + R_2C_2S + 1}{C_1S(R_1C_1S + 1)}\right] \cdot I(S)$$

$$\text{or,} \quad E_I(S) = \left[\frac{1 + R_1C_2S + R_1C_2S + R_2C_2S + R_1R_2C_1C_2S^2}{C_1S(R_1C_1S + 1)}\right] \cdot I(S)$$

Again,

$$E_0(S) = Z_2 \cdot I(S) = \left[\frac{R_2C_2S + 1}{C_2S}\right] \cdot I(S)$$

Dividing equation **(2)** by **(1)**; we get,

$$\frac{E_0(S)}{E_I(S)} = \frac{\left[\frac{R_2C_2S + 1}{C_2S}\right] \cdot I(S)}{\left[\frac{1 + R_1C_1S + R_1C_2S + R_2C_2S + R_1R_2C_1C_2S^2}{C_2S(1 + R_1C_1S)}\right] \cdot I(S)}$$

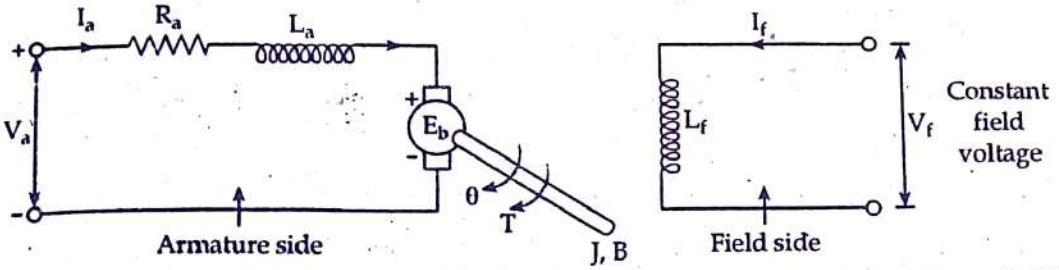
$$\text{or,} \quad \frac{E_0(S)}{E_I(S)} = \frac{(R_2C_2S + 1)(R_1C_1S + 1)}{1 + R_1C_1S + R_2C_2S + R_2C_2S + R_1R_2C_1C_2S^2}$$

$$\therefore \quad \boxed{\frac{E_0(S)}{E_I(S)} = \frac{(1 + R_1C_1S)(1 + R_2C_2S)}{R_1C_2S + (1 + R_1C_1S)(1 + R_2C_2S)}}$$

Scanned with CamScanner

2.3.4 DC–MOTOR

2.3.4.1 Armature Control Method for Speed Control of DC–Motor (Closed Loop Control System)



Consider the armature controlled dc motor. The field parameters are constant. Since, field current I_f is constant, the flux ϕ is also constant. There is constant field voltage V_f.

Let, R_a = Armature resistance

L_a = Armature self inductance

I_a = Armature current

I_f = Field current

E_b = Induced e.m.f. in armature (back e.m.f.)

V_a = Applied voltage in armature

V_f = Voltage in field

T = Torque development or produced by the motor

θ = Angular displacement of the motor shaft

J = Equivalent moment of inertia of motor shaft

B = Equivalent coefficient of friction of motor

i) Electrical equations

Applying KVL in the armature circuit;

$$V_a = I_a R_a + L_a \frac{dI_a}{dt} + E_b \quad (1)$$

When armature is rotating, e.m.f. is induced.

$$E_b \propto \phi\omega$$

$$E_b = K_b\omega \text{ [since, } \phi \text{ is constant]}$$

$$E_b = K_b \frac{d\theta}{dt} \quad (2)$$

where, K_b is the back e.m.f. constant.

ω is the angular velocity $= \left(\frac{d\theta}{dt}\right)$.

θ = is the angular displacement.

ii) Mechanical equations

The equation with MOI and coefficient of friction is;

$$T = J\left(\frac{d^2\theta}{dt^2}\right) + B\left(\frac{d\theta}{dt}\right) \quad (3)$$

Torque 'T' of the motor is the product of armature current and flux.

$$T \propto \phi I_a$$

since, flux ϕ is constant.

$$T = KI_a \quad (4)$$

where, K is the motor torque constant.

Taking L.T. of equation (1); we have,

Scanned with CamScanner

$$V_a(S) = R_aI_a(S) + SL_aI_a(S) + E_b(S)$$
$$= (R_a + SL_a)I_a(S) + E_b(S)$$

Taking L.T. of equation (2); we have,

$$E_b(S) = SK_b\theta(S) = K_bS\theta(S)$$

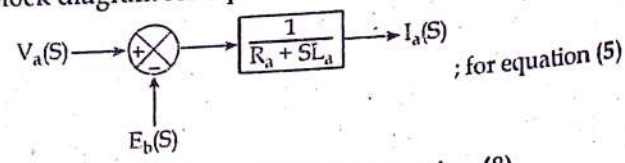
Taking L.T. of equation (3); we have,

$$T(S) = S^2J\theta(S) + SB\theta(S) = (S^2J + SB)\theta(S) = (SJ + B)S\theta(S)$$

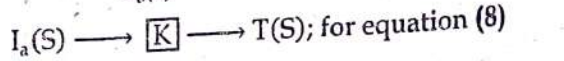
Taking L.T. of equation (4); we have,

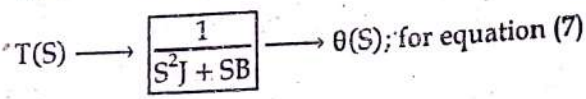
$$T(S) = KI_a(S)$$

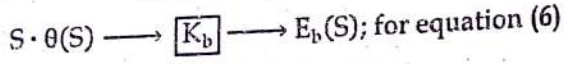
The block diagram for equation (5), (6), (7) and (8) is;



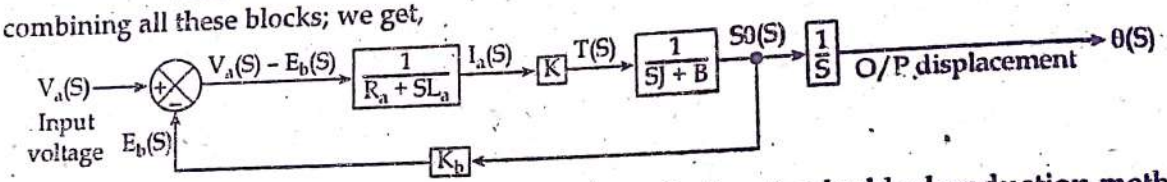
; for equation (5)



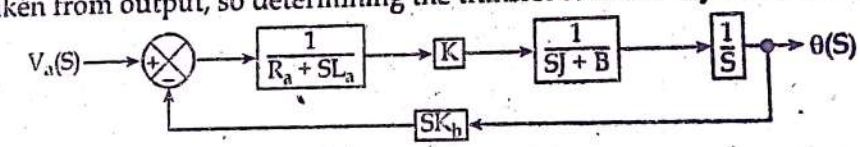


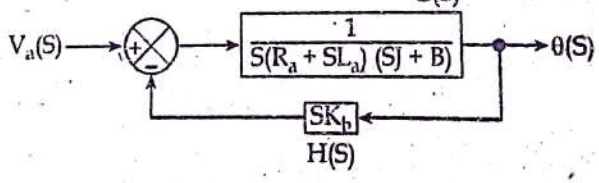


Now, combining all these blocks; we get,



The feedback is taken from output, so determining the transfer function by block reduction method.





Here,

$$\text{Forward T.F. } G(S) = \frac{K}{S(R_a + SL_a)(SJ + B)}$$

$$\text{Backward T.F. } H(S) = SK_b$$

and, $$\text{T.F.} = \frac{\theta(S)}{V_a(S)} = \frac{G(S)}{1 + G(S)H(S)}$$

The block diagram shows feedback T.F. $H(S) = SK_b$; so that, the "armature method for DC motor" to control the speed is closed loop control system.

We get,

$$V_a(S) = [R_a + SL_a]I_a(S) + E_b(S) \quad (5)$$

$$E_b(S) = K_bS \cdot \theta(S) \quad (6)$$

$$T(S) = [SJ + B] \cdot S\theta(S) \quad (7)$$

Scanned with CamScanner

and, $T(S) = K \cdot I_a(S)$ (8)

so, $V_a(S) = [R_a + SL_a]I_a(S) + E_b(S)$

$= [R_a + SL_a] \cdot \frac{T(S)}{K} + K_b \cdot S\theta(S)$; from equation (8) and (6)

$= [R_a + SL_a] \cdot \frac{[SJ + B] \cdot S\theta(S)}{K} + K_b \cdot S\theta(S)$; from equation (7)

or, $V_a(S) = \frac{(R_a + SL_a)[SJ + B] \cdot S\theta(S) + KK_b \cdot S\theta(S)}{K} = \frac{\theta(S)[(R_a + SL_a)(SJ + B) \cdot S + SKK_b]}{K}$

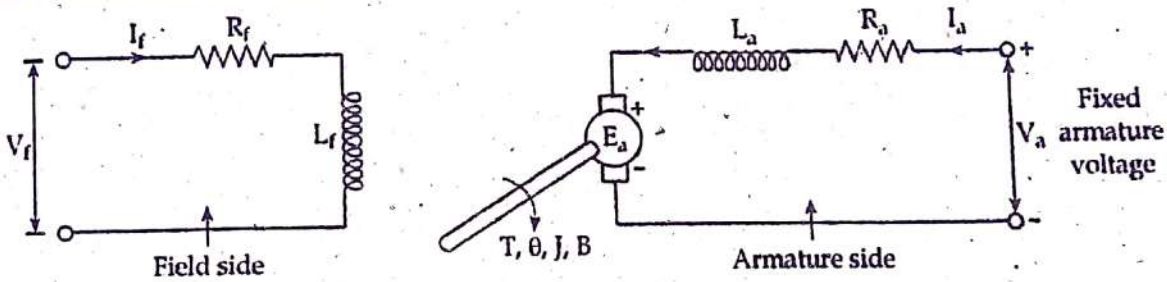
or, $\frac{\theta(S)}{V_a(S)} = \frac{K}{S(R_a + SL_a)(SJ + B) + SKK_b} = \frac{K}{SR_a\left(1 + S\frac{L_a}{R_a}\right)B\left(1 + S\frac{J}{B}\right) + SKK_b}$

$\therefore$ $\boxed{\frac{\theta(S)}{V_a(S)} = \frac{K}{SR_aB(1 + S\tau_a)(1 + S\tau_b) + SKK_b}}$; which is the required transform function.

where, $\tau_a = \frac{L_a}{R_a}$ = Time constant of armature ckt.

$\tau_b = \frac{J}{B}$ = Mechanical time constant.

2.3.4.2 Field Control Method for Speed Control of DC–Motor (Open Loop Control System)



Applying K.V.L. in the field;

$V_f = I_fR_f + L_f\frac{dI_f}{dt}$ (1)

and, constant I_a (armature) current is fed to the armature circuit. The flux ϕ is proportional to the field current.

$\phi \propto I_f$

or, $\phi = K_fI_f$ (2)

where, K_f is the field constant.

and, torque developed by the motor is proportional to the flux and armature current.

$T \propto I_a\phi$

or, $T = K'\phi I_a$

Substituting the value of ϕ; we get,

$T = K'K_fI_fI_a$

or, $T = KK_fI_f$ (3)

where, $K = K'I_a$.

and, Torque $(T) = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt}$ (4)

Taking L.T. of the equation (1); we get,

$V_f(S) = R_fI_f(S) + L_fSI_f(S) = [R_f + SL_f]I_f(S)$ (5)

Taking L.T. of the equation (3); we get,

$T(S) = KK_fI_f(S)$ (6)

Scanned with CamScanner

Taking L.T. of the equation (4); we get,

$$T(S) = S^2J\theta(S) + SB\theta(S) = S[SJ + B]\theta(S)$$

From the equation (5) and (6); we have,

$$V_f(S) = (R_f + SL_f)I_f(S) = (R_f + SL_f)\frac{T(S)}{KK_f}$$

Substituting the value of equation (7); we have,

$$V_f(S) = (R_f + SL_f)\frac{S(SJ + B)}{KK_f}\theta(S)$$

or, $$\frac{\theta(S)}{V_f(S)} = \frac{KK_f}{S(R_f + SL_f)(SJ + B)}$$

or, $$\frac{\theta(S)}{V_f(S)} = \frac{KK_f}{SR_f\left(1 + S\frac{L_f}{R_f}\right)B\left(1 + S\frac{J}{B}\right)}$$

$$\therefore \quad \boxed{\frac{\theta(S)}{V_f(S)} = \frac{KK_f}{SR_fB(1 + S\tau_a)(1 + S\tau_b)}}$$

which is the required expression for transfer function.

where, $\tau_a = \frac{L_f}{R_f}$ = Time constant of field circuit.

$\tau_b = \frac{J}{B}$ = Mechanical time constant.

From equation (5), (6) and (7); we have,

$$V_f(S) \longrightarrow \boxed{\frac{1}{R_f + SL_f}} \xrightarrow{I_f(S)} \boxed{KK_f} \xrightarrow{T(S)} \boxed{\frac{1}{S(SJ + B)}} \longrightarrow \theta(S)$$

Reducing the block; we have,

$$V_f(S) \longrightarrow \boxed{\frac{KK_f}{S(R_f + SL_f)(SJ + B)}} \longrightarrow \theta(S)$$

In terms of time constant;

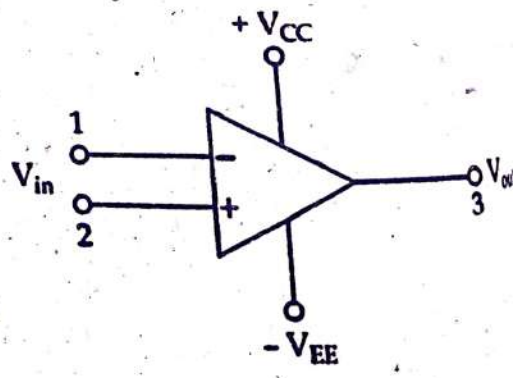
$$V_f(S) \longrightarrow \boxed{\frac{KK_f}{SR_fB(1 + S\tau_a)(1 + S\tau_b)}} \longrightarrow \theta(S)$$

Since, no feedback, the field control method to control speed of DC-motor is **open loop control system.**

2.3.5 Operational Amplifier (OP-Amp)

There are manly five terminals in OP-Amp.

i) Inverting input terminal
ii) Non inverting input terminal
iii) Output terminal
iv) Positive supply terminal
v) Negative supply terminal



If V_d is the differential voltage applied to the ideal OP-Amp then output of an amplifier is;

$$V_{out} = A \times V_d$$

where, A is the amplifiers gain $\approx 10^6$.

The properties of ideal OP-Amp are;

i) Its open loop gain is infinite. (*i.e.*, $A = \infty$)
ii) Input impedance is infinite (*i.e.*, $Z_{in} = \infty$)
iii) Output impedance is zero. ($Z_{out} = 0$)

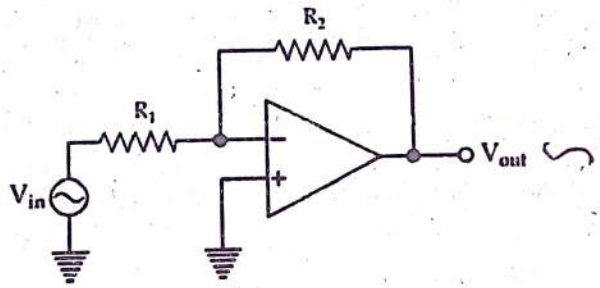
Scanned with CamScanner

iv) It has infinite bandwidth.
v) It has zero common mode gain or infinite common mode rejection
vi) The two input terminals are virtually grounded.
vii) Common mode rejection ratio is infinite [CMRR → ∞]
viii) Slew rate is infinite. [SR → ∞]

OP-Amp is used as:

i) Inverting amplifier
ii) Non-inverting amplifier
iii) Adder or summer
iv) Subtractor
v) Integrator
iv) Differentiator
vii) Wave generator (square, triangular, sinusoidal wave)
viii) Amplifier

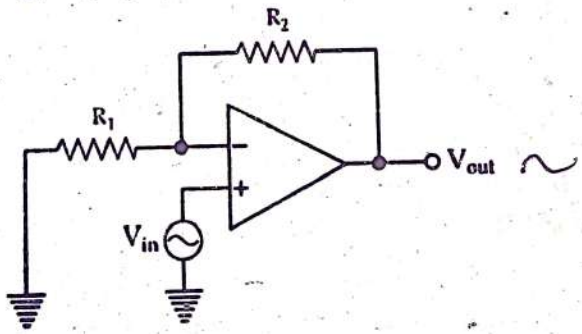
i) OP-Amp as inverting amplifier



$$V_{out} = -\left(\frac{R_2}{R_1}\right) V_{in}$$

Negative sign indicates that o/p signal is out of phase with i/p signal.

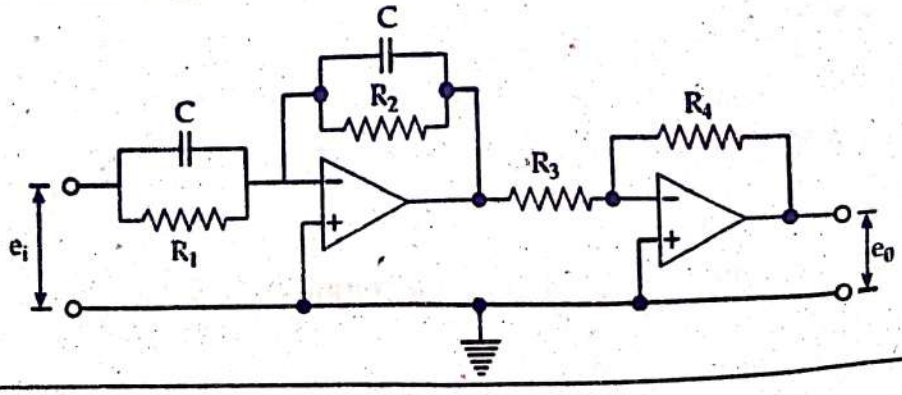
ii) OP-Amp as Non-inverting amplifier



$$V_{out} = \left(1 + \frac{R_2}{R_1}\right) V_{in}$$

EXAMPLE 2.12

Obtain the transfer function of given network.



Scanned with CamScanner

Comparing equation (2) and (3); we have,

	Mechanical system	Electoral system
1.	Force (F)	Voltage (V)
2.	Mass (M)	Inductance (L)
3.	Damping coefficient (B)	Resistance (R)
4.	Stiffness (K)	Reciprocal of capacitance $\left(\frac{1}{C}\right)$
5.	Displacement (x)	Charge (q)
6.	Velocity (V)	Current (I)

Reciprocal of stiffness is called "Elastance"

Reciprocal of resistance is called "conductance"

$$\text{Elastance} = \left(\frac{1}{K}\right)$$

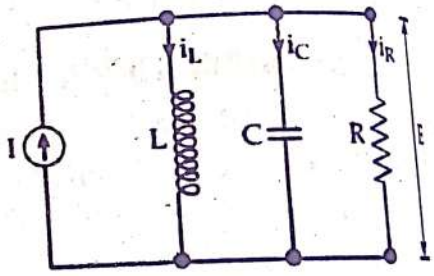
and, $\text{Conductance (G)} = \left(\frac{1}{R}\right)$

Statement:

"Each junction in the mechanical system corresponds to a closed loop which consists of electrical excitation sources and passive elements analogous to the mechanical driving sources and passive elements connected to the junction. All points on a rigid mass are considered as junction."

2.4.2 Force–Current (F–I) Analogy

Consider a LCR parallel circuit with current source 'I' as shown in the given circuit below;



(1)

Applying KCL; we have,

$$I = i_L + i_C + i_R = \frac{1}{L}\int E\,dt + C\frac{dE}{dt} + \frac{E}{R}$$

Since, L, C, R are in parallel; so, equal voltage (E).

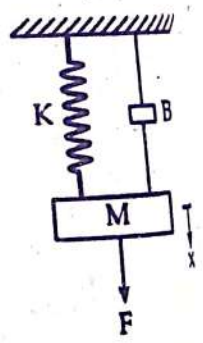
If ϕ is the flux linkage of inductor; then,

$$E = \frac{d\phi}{dt}; \text{ from Faraday's law}$$

so, $$I = \left(\frac{1}{L}\right)\phi + C\left(\frac{d^2\phi}{dt^2}\right) + \frac{1}{R}\left(\frac{d\phi}{dt}\right)$$

or, $$I = C\left(\frac{d^2\phi}{dt^2}\right) + \frac{1}{R}\left(\frac{d\phi}{dt}\right) + \left(\frac{1}{L}\right)\phi$$

and, if we consider a mechanical system as shown aside; then,



(2)

(3)

$$F = M\left(\frac{d^2x}{dt^2}\right) + B\left(\frac{dx}{dt}\right) + Kx$$

Comparing equation (2) and (3); we get,

	Mechanical translational system	Electrical system
1.	Force (F)	Current (I)
2.	Mass (M)	Capacitance (c)
3.	Damping coefficient (B)	Conductance $\left(G = \frac{1}{R}\right)$
4.	Stiffness (K)	Reciprocal of inductance $\left(\frac{1}{L}\right)$
5.	Displacement (x)	Flux linkage (ϕ)
6.	Velocity (V) or $\left(\frac{dx}{dt}\right)$	Voltage (E) i.e., $\frac{d\phi}{dt}$

Statement:
"Each junct
and passiv
junction. A
analogous
Capacitan
displacem

EXAMPLE
Draw the

Solutio
Now, d
and the

Now,

EXA
Dra

Sc
H
H
N

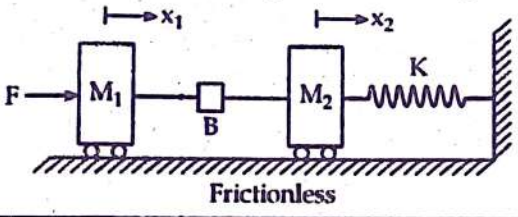
Scanned with CamScanner

Statement:
"Each junction in the mechanical system corresponds to a node which joins electrical excitation sources and passive elements analogous to mechanical driving sources and passive elements connected to the junction. All points on a rigid mass are considered as same junction and one terminal of capacitance analogous to a mass is always connected to ground."

Capacitance analogous to mass is always connected to the ground because that the velocity (or displacement), of a mass is always referred to earth.

EXAMPLE 2.13

Draw the F–V and F–I analogous circuit of following mechanical system.



Solution:

Now, drawing F–V analogous system; here, 2 masses, and 2 displacement (x_1 and x_2); so, two junctions and there will be 2 loops.

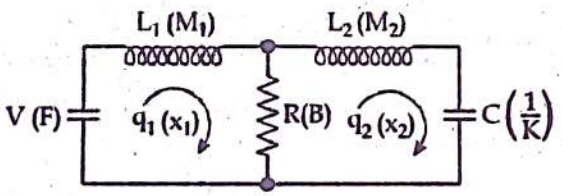


Figure: F–V analogous ckt

Now, drawing F–I analogous system; we have,

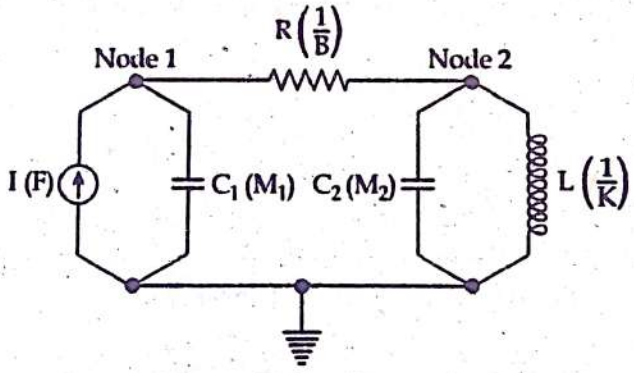
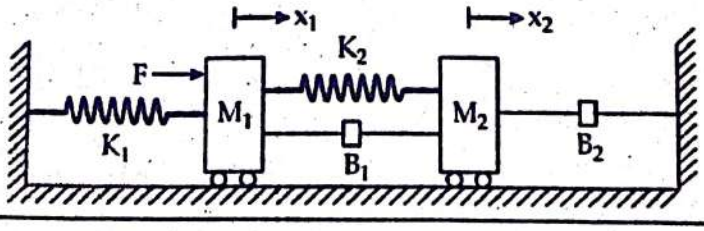


Figure: F–I analogous ckt

EXAMPLE 2.14

Draw the F–V and F–I analogous circuit of following mechanical system.



Solution:

Here, two masses (M_1 and M_2), and two displacement (x_1 and x_2); so, two junctions.
Hence, two closed loops [refer figure (i)].
Now, drawing F–I analogous system [refer figure (ii)]; we have,

Scanned with CamScanner

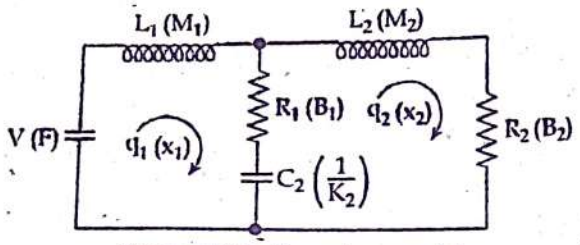


Figure: (i) F–V analogous ckt

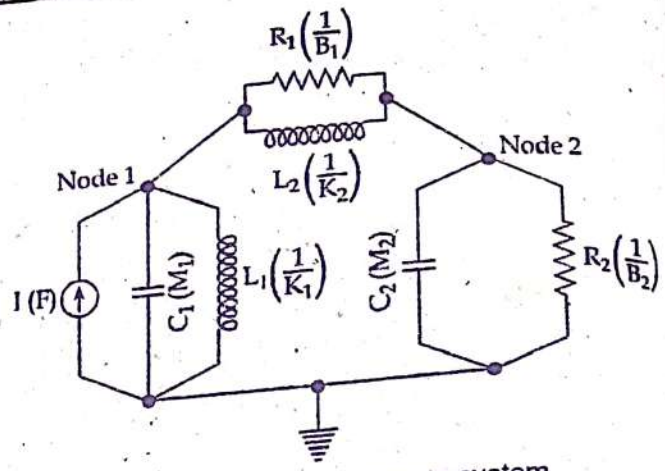
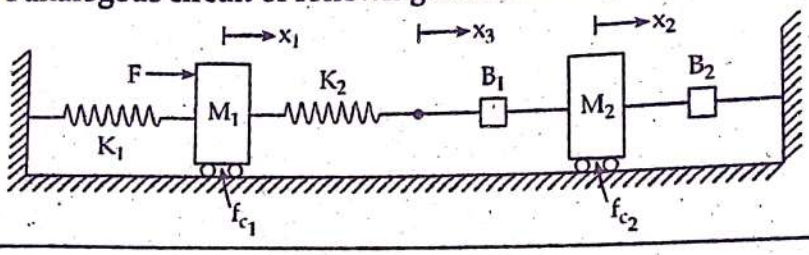


Figure: (ii) F–I analogous system

EXAMPLE 2.15

Draw the F–V and F–I analogous circuit of following mechanical system.



Solution:

Here, three masses including mass-less body (M_0); so, there are three displacements (x_1, x_2 and x_3); so, there will be three loops.

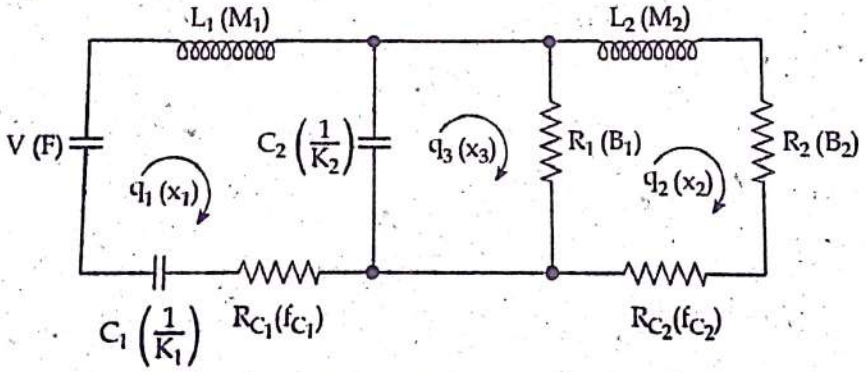


Figure: F–V analogous circuit

- Mass-less mass $M_0 = 0$; so, we do not consider inductance (L_0) as $L_0 = 0$.
- Effect of frictional constant is same as that of damping coefficient (B).

Now, drawing F–I analogous system; we have,

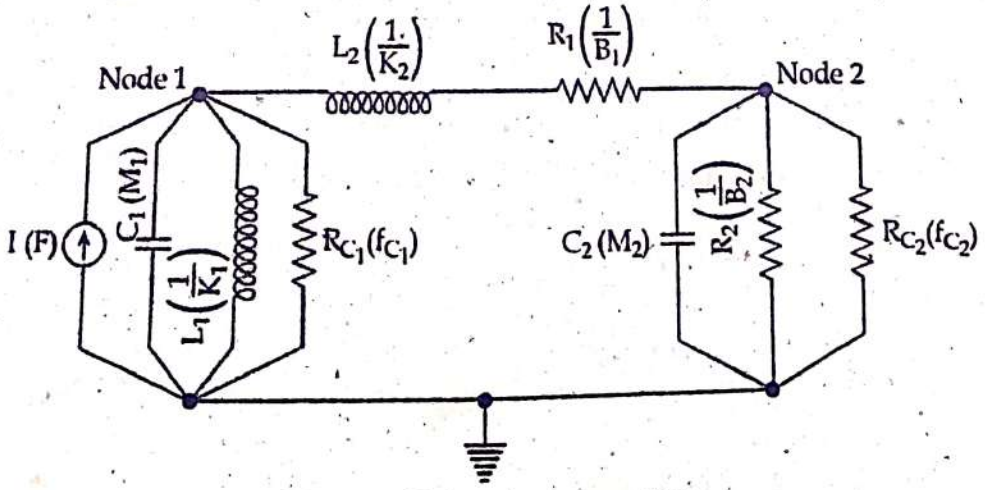
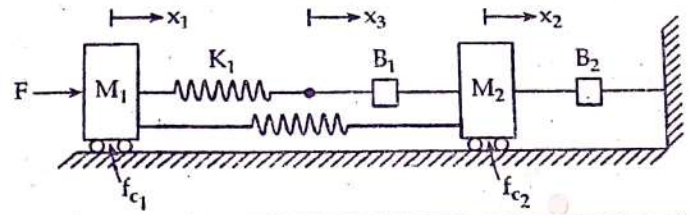


Figure: F–I analogous system

Scanned with CamScanner

EXAMPLE 2.16

Draw the F–V and F–I analogous circuit of following mechanical system.



Solution:

Three masses M_1, M_2 and mass-less mass (M_0). Three displacements x_1, x_2 and x_3; so, there is three junctions and said to have three loops in the F–V analogous circuit.

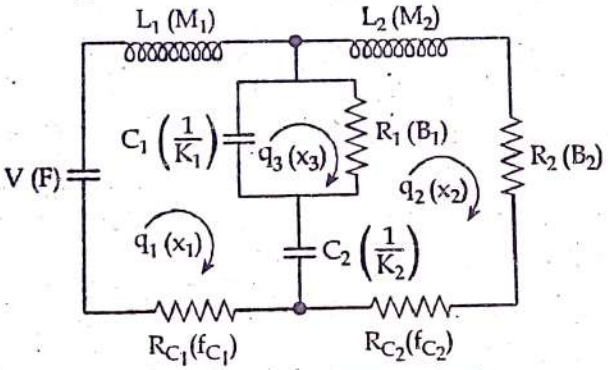


Figure: F–V analogous circuit

In F-V analogous circuit

NOTE

i) If there are two different Restoring forces serially then we consider mass–less mass having $M_0 = 0$.
ii) Number of mass = Number of displacement = Number of loops
iii) If B and K are in series, then R and C are in parallel as in 2.19 and 2.20.
iv) If B and K are in parallel, then R and C are in series as in 2.18 and 2.20.

Now, drawing F–I analogous system; we have,

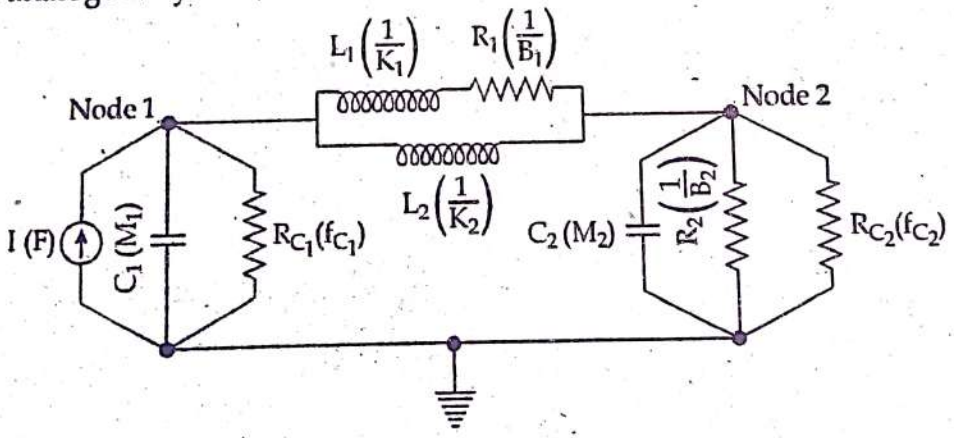
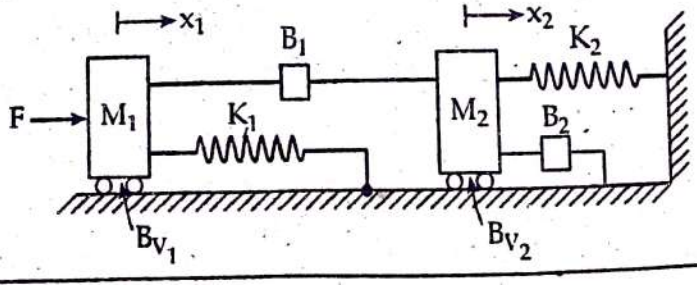


Figure: F–I analogous system

EXAMPLE 2.17

Draw the F–V and F–I analogous circuit of the following mechanical system.



Scanned with CamScanner

Solution:

The F–I analogy of the above circuit is;

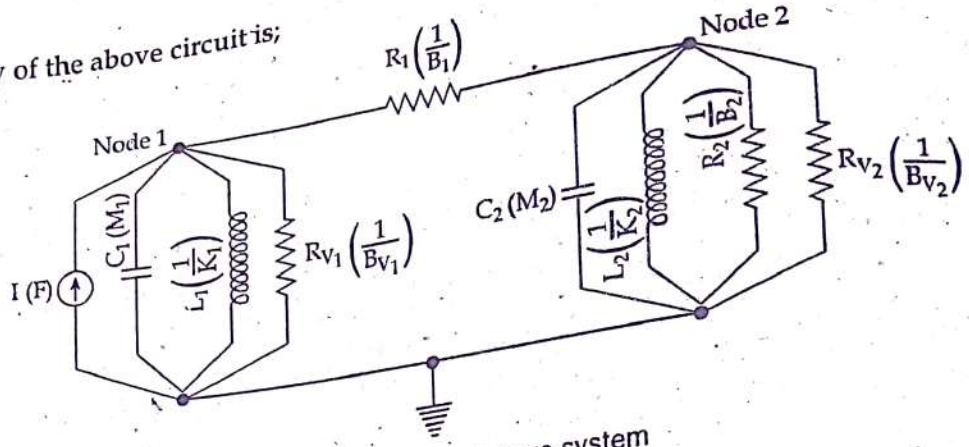


Figure: F–I analogous system

F–V analogy of the circuit above is;

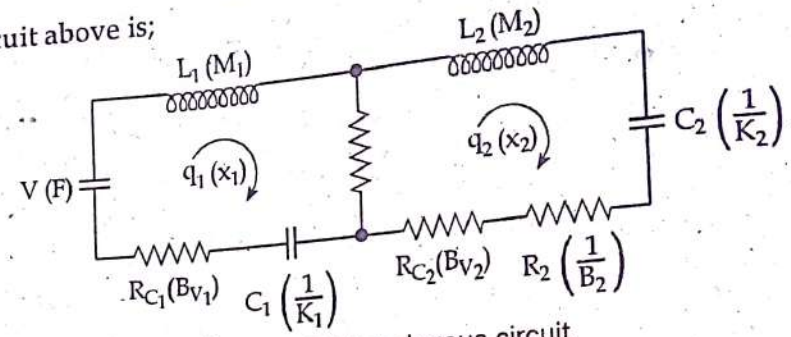


Figure: F–V analogous circuit

In F–I analogous circuit

NOTE

i) Capacitance is always grounded.
ii) If B and K are in parallel, the R and L are also parallel.
iii) If B and K are in series, the R and L are also in series

2.4.3 Torque Voltage Analogy: (T–V) Analogy

	Mechanical rotational system	Electrical system
1.	Torque (T)	Voltage (V)
2.	Moment of Inertia (J)	Inductance (L)
3.	Damping coefficient (B)	Resistance (R)
4.	Stiffness (K)	Reciprocal of capacitance $\left(\frac{1}{C}\right)$
5.	Angular displacement (θ)	Charge (q)
6.	Angular velocity (ω)	Current (I)

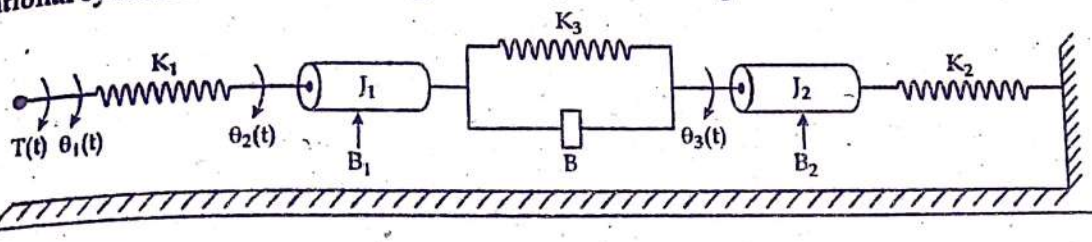
2.4.4 Torque Current Analogy: (T–I) Analog

	Mechanical rotational system	Electrical system
1.	Torque (T)	Current (I)
2.	Moment of inertia (J)	Capacitance (C)
3.	Damping coefficient (B)	Reciprocal of resistance $\left(\frac{1}{R}\right)$
4.	Stiffness (K)	Reciprocal of inductance $\left(\frac{1}{L}\right)$
5.	Angular displacement (θ)	Flux linkage (ϕ)

Scanned with CamScanner

EXAMPLE 2.18

For a rotational system as shown in the figure below; draw analogous circuit based on T–I analogy.



Solution:

Drawing a electrical equivalent circuit, the T–I analogy is;

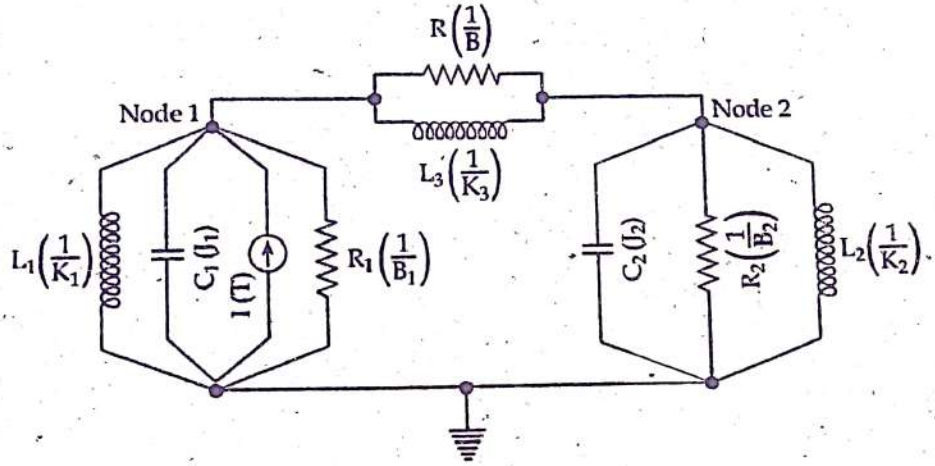
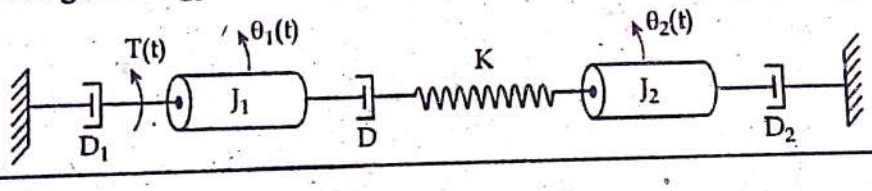


Figure: T–I analogous circuit

EXAMPLE 2.19

Develop a torque–voltage analogy circuit. **[2072 Kartik]**



Solution:

The T–V analogous circuit be;

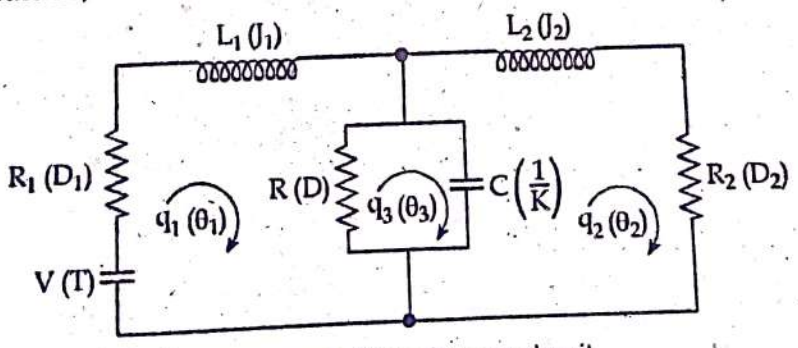
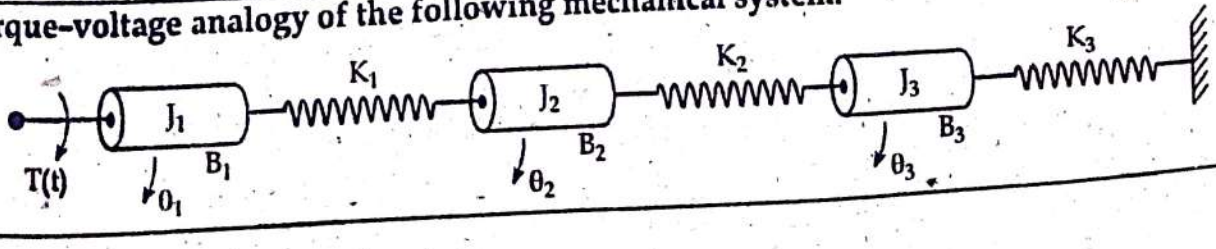


Figure: T–V analogous circuit

EXAMPLE 2.20

Draw torque–voltage analogy of the following mechanical system. **[2068 Baishakh]**



Scanned with CamScanner

Solution:

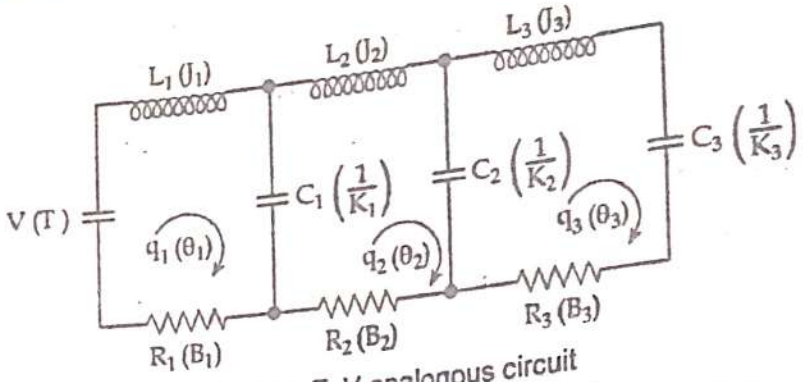


Figure: T–V analogous circuit

2.5 LINEARIZED APPROXIMATION OF NON–LINEAR CHARACTERSTICS

This is used to obtain a linear relation (or Mathematical model) for a non–linear system. So, we assume that the variable deviate only slightly from some operating condition.

Linear system

A system where the superposition theorem can be implemented *i.e.*, response produced by simultaneous application of two different inputs is the sum of two individual responses.

If $y_1(t)$ and $y_2(t)$ are two corresponding outputs for two input signals $x_1(t)$ and $x_2(t)$, and $x_3(t)$ is applied such that;

$$x_3(t) = ax_1(t) + bx_2(t)$$

Then, the corresponding output is $y_3(t)$.

The system will be linear if and only if;

$$y_3(t) = ay_1(t) + by_2(t)$$

where, a and b are constants.

i) Approximation for single input

Consider a non–linear system with input x(t) and output z(t) such that output is related to input as;

$$z = f(x) \quad (1)$$

Let, $\bar{x}$ and $\bar{z}$ be the equilibrium points (normal operating points of the system).

As $x = \bar{x}$ and $z = \bar{z}$, the variation in variables in very small, so expanding equation (1) according to Taylor's series.

$$z = f(\bar{x}) + \frac{df(\bar{x})}{dx}(x - \bar{x}) + \frac{1}{2}\frac{d^2f(\bar{x})}{dx^2}(x - \bar{x})^2 + \ldots\ldots$$

As $(x - \bar{x})$ is very small; so, higher order of $(x - \bar{x})$ can be neglected;

$$z = f(\bar{x}) + \frac{df(\bar{x})}{dx}(x - \bar{x}) \quad (2)$$

We know that;

$$\bar{z} = f(\bar{x})$$

so, equation (2) becomes;

$$z = \bar{z} + K(x - \bar{x})$$

$$(z - \bar{z}) = K(x - \bar{x})$$

where, $K = \frac{df(\bar{x})}{dx}$.

$\therefore$ $\boxed{(z - \bar{z}) = K(x - \bar{x})}$;

which is the linearized approximation of equation (1).

Scanned with CamScanner

EXAMPLE 2.21

Linearize: $z = x^2$ at $x = 6$.

Solution:

Here, z is non linear and function of x.

The operating point given is at $x = 6$.

so, $z = f(x) = x^2$

and, $\bar{x} = 6$

To linearize the non linear system; we have,

$$(z - \bar{z}) = K(x - \bar{x})$$

where, $K = \frac{df(\bar{x})}{dx} = \frac{d\left[(\bar{x})^2\right]}{dx} = 2\bar{x} = 2 \times 6 = 12.$

$$\bar{z} = f(\bar{x}) = (\bar{x})^2 = (6)^2 = 36.$$

so, $(z - \bar{z}) = K(x - \bar{x})$ becomes;

$$(z - 36) = 12(x - 6)$$

or, $z - 36 = 12x - 72$

or, $z = 12x - 36$

ii) Approximation for multiple input

Consider a non-linear system with inputs x(t) and y(t) and output is z(t) such that output is related to its inputs as;

$$z = f(x, y) \quad \text{(1)}$$

Let, $\bar{x}$, $\bar{y}$ and $\bar{z}$ be the equilibrium points (normal operating points) of the system.

As $x = \bar{x}$, $y = \bar{y}$ and $z = \bar{z}$; the variation in variables is very small so expanding equation (1) according to Taylor's series; we get,

$$z = f(\bar{x}, \bar{y}) + \left[(x - \bar{x})\frac{\partial f(\bar{x}, \bar{y})}{\partial x} + (y - \bar{y})\frac{\partial f(\bar{x}, \bar{y})}{\partial y}\right] + \frac{1}{2!}\left[(x - \bar{x})^2\frac{\partial^2 f(\bar{x}, \bar{y})}{\partial x^2} + (y - \bar{y})^2\frac{\partial^2 f(\bar{x}, \bar{y})}{\partial y^2}\right] + \ldots\ldots$$

As $(x - \bar{x})$, $(y - \bar{y})$ are very small so neglecting higher power terms;

$$z = f(\bar{x}, \bar{y}) + (x - \bar{x})\frac{\partial f(\bar{x}, \bar{y})}{\partial x} + (y - \bar{y})\frac{\partial f(\bar{x}, \bar{y})}{\partial y}$$

and, we know that;

$$\bar{z} = f(\bar{x}, \bar{y})$$

so, equation (1) becomes;

$$z = \bar{z} + (x - \bar{x})K_1 + (y - \bar{y})K_2$$

$$\therefore \quad \boxed{(z - \bar{z}) = K_1(x - \bar{x}) + K_2(y - \bar{y})};$$

which is the linearized approximation of equation (1).

where, $K_1 = \frac{\partial f(\bar{x}, \bar{y})}{\partial x}$ and $K_2 = \frac{\partial f(\bar{x}, \bar{y})}{\partial y}$.

EXAMPLE 2.22

Linearize the non-linear system $Z = x^2 + 4xy + 6y^2$ in the region defined as $8 \le x \le 10$ and $2 \le y \le 4$. Find the error if the linearized equation is used to calculate value of z when $x = 8$ and $y = 3$.

Solution:

Here, $Z = x^2 + 4xy + 6y^2$ is a non-linear system having two inputs x and y.

Scanned with CamScanner

The operating points are;

$$\bar{x} = \left(\frac{8+10}{2}\right) = 9$$

$$\bar{y} = \left(\frac{2+4}{2}\right) = 3$$

To linearize the non-linear system; we have,

$$(z - \bar{z}) = K_1(x - \bar{x}) + K_2(y - \bar{y})$$

where, $\bar{z} = f(\bar{x}, \bar{y}) = (\bar{x})^2 + 4(\bar{x})(\bar{y}) + 6(\bar{y})^2 = (9)^2 + 4 \times 9 \times 3 + 6 \times (3)^2 = 243.$

and, $K_1 = \dfrac{\partial f(\bar{x}, \bar{y})}{\partial x} = \dfrac{\partial\left[(\bar{x})^2 + 4(\bar{x})(\bar{y}) + 6(\bar{y})^2\right]}{\partial x} = 2\bar{x} + 4\bar{y} = 2 \times 9 + 4 \times 3 = 30$

Similarly;

$$K_2 = \frac{\partial f(\bar{x}, \bar{y})}{\partial y} = \frac{\partial\left[(\bar{x})^2 + 4(\bar{x})(\bar{y}) + 6(\bar{y})^2\right]}{\partial y} = 4\bar{x} + 6 \times 2\bar{y} = 4 \times 9 + 6 \times 2 \times 3 = 72$$

Therefore, the approximized form is;

$$(z - \bar{z}) = K_1(x - \bar{x}) + K_2(y - \bar{y})$$

or, $z - 243 = 30(x - 9) + 72(y - 3) = 30x - 270 + 72y - 216$

$\therefore$ $z = 30x + 72y - 243$; which is the required linear system.

To find error

Value of Z in non linear system when x = 8 and y = 3

$$Z_N = x^2 + 4xy + 6y^2 = (8)^2 + 4 \times 8 \times 3 + 6 \times (3)^2 = 214$$

Value of Z in linear system when x = 8 and y = 3

$$Z_L = 30x + 72y - 243 = 30 \times 8 + 72 \times 3 - 243 = 213$$

$\therefore$ $$\text{Error} = \left|\frac{Z_{\text{non-linear}} - Z_{\text{linear}}}{Z_{\text{non-linear}}}\right| \times 100\%$$

$$= \left|\frac{Z_N - Z_L}{Z_N}\right| \times 100\%$$

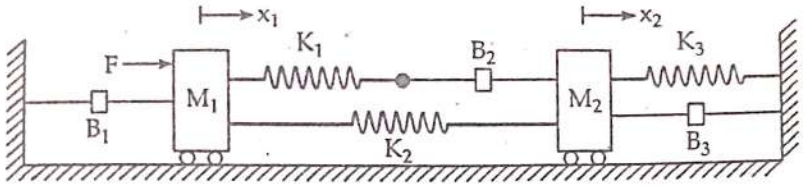
$$= \left|\frac{214 - 213}{214}\right| \times 100\%$$

$$= 0.4673\%$$

Scanned with CamScanner

2.6 SOLVED PROBLEMS

1. Write mathematical equations and draw a free body diagram of circuit below.



Solution:

Let, M_0 be the mass of mass-less body and x_0 as displacement F.B.D. of M_1 is;

$$F \longrightarrow \boxed{M_1} \begin{cases} \longleftarrow K_1(x_1 - x_0) \\ \longleftarrow K_2(x_1 - x_2) \\ \longleftarrow M_1\ddot{x}_1 \end{cases}$$
$$B_1\dot{x}_1 \longleftarrow$$

$$F = B_1\dot{x}_1 + K_1(x_1 - x_0) + K_2(x_1 - x_2) + M_1\ddot{x}_1$$

It can be written as;

$$F = B_1\frac{dx_1}{dt} + M_1\frac{d^2x_1}{dt^2} + (K_1 + K_2)x_1 - K_1x_0 - K_2x_2 \quad (1)$$

Now, F.B.D. of M_0 is;

$$K_1(x_1 - x_0) \longrightarrow \boxed{M_0} \longleftarrow B_2(\dot{x}_0 - \dot{x}_2)$$

and, $K_1(x_1 - x_0) = B_2(\dot{x}_0 - \dot{x}_2)$

or, $K_1(x_1 - x_0) = B_2\frac{d}{dt}(x_0 - x_2)$ (2)

Again, F.B.D. of M_2 is;

$$B_2(\dot{x}_0 - \dot{x}_2) \longrightarrow \boxed{M_1} \begin{cases} \longleftarrow K_3x_2 \\ \longleftarrow B_3\dot{x}_2 \\ \longleftarrow M_2\ddot{x}_2 \end{cases}$$
$$K_2(x_1 - x_2) \longrightarrow$$

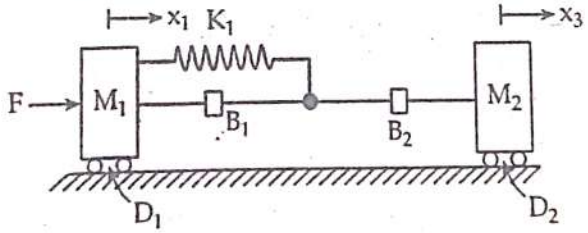
$$B_2(\dot{x}_0 - \dot{x}_2) + K_2(x_1 - x_2) = K_3x_2 + B_3\dot{x}_2 + M_2\ddot{x}_2$$

or, $B_2\dot{x}_0 + K_2x_1 = B_2\dot{x}_2 + K_2x_2 + K_3x_2 + B_3\dot{x}_2 + M_2\ddot{x}_2$

or, $K_2x_1 + B_2\frac{dx_0}{dt} = (K_2 + K_3)x_2 + (B_2 + B_3)\frac{dx_2}{dt} + M_2\frac{d^2x_2}{dt^2}$ (3)

Equations (1), (2) and (3) are the required differential equations.

2. Draw the free body diagram of the circuit shown below.



Solution:

Here, we have to consider mass-less Mass (M_0) and having displacement x_0.

F.B.D. of M_1 is;

$$F \longrightarrow \boxed{M_1} \begin{cases} \longleftarrow K_1(x_1 - x_0) \\ \longleftarrow B_1(\dot{x}_1 - \dot{x}_0) \\ \longleftarrow D_1\dot{x}_1 \\ \longleftarrow M_1\ddot{x}_1 \end{cases}$$

$$F = K_1(x_1 - x_0) + B_1(\dot{x}_1 - \dot{x}_0) + D_1\dot{x}_1 + M_1\ddot{x}_1$$

Scanned with CamScanner

F.B.D. of M_0 is;

$K_1(x_1 - x_0) \longrightarrow$ [M_0] $\longleftarrow B_2(\dot{x}_0 - \dot{x}_3)$

$B_1(\dot{x}_1 - \dot{x}_0) \longrightarrow$

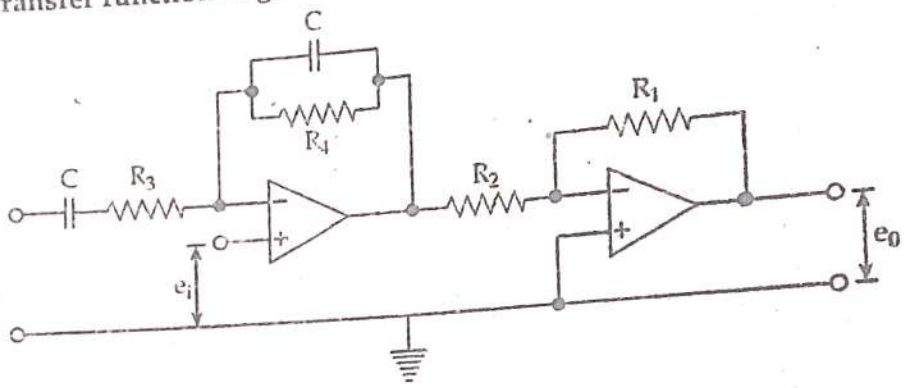
$$K_1(x_1 - x_0) + B_1(\dot{x}_1 - \dot{x}_0) = B_2(\dot{x}_0 - \dot{x}_3)$$

F.B.D. of M_2 is;

$B_2(\dot{x}_0 - \dot{x}_3) \longrightarrow$ [M_2] $\longleftarrow M_2\ddot{x}_3$, $\longleftarrow D_2\dot{x}_3$

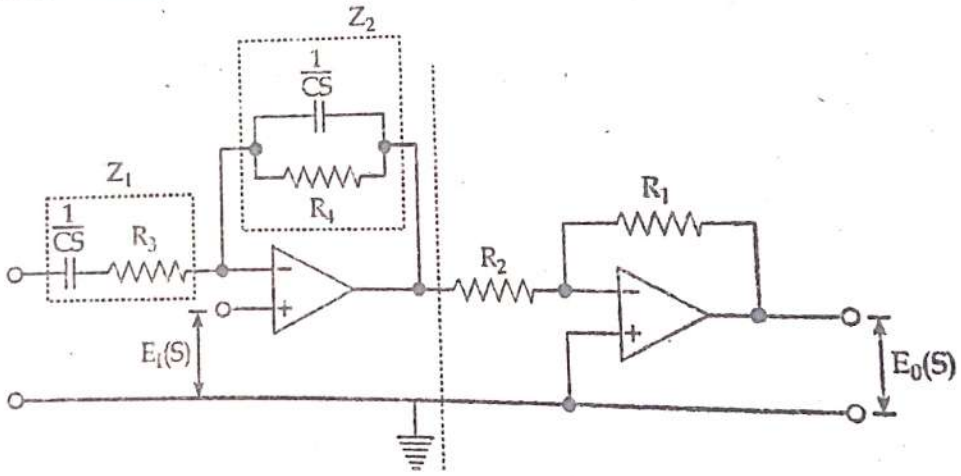
$$B_2(\dot{x}_0 - \dot{x}_3) = M_2\ddot{x}_3 + D_2\dot{x}_3$$

3. **Obtain the transfer function of given network.**



Solution:

The circuit in S-domain becomes;



From the circuit; we have,

R_3 and C are in series;

so, Combined impedance $(Z_1) = \left(R_3 + \frac{1}{CS}\right) = \frac{1 + R_3CS}{CS}$

and, R_4 and C are in parallel;

$$\text{Combined impedance } (Z_2) = \frac{R_4\left(\frac{1}{CS}\right)}{R_4 + \frac{1}{CS}} = \frac{R_4}{1 + R_4CS}$$

The output of non-inverting amplifier of section I is given by $E_1(S)$.

$$E_1(S) = \left[1 + \frac{Z_2(S)}{Z_1(S)}\right] E_1(S)$$

$$\frac{E_1(S)}{E_1(S)} = \left[1 + \frac{Z_2(S)}{Z_1(S)}\right]$$

Scanned with CamScanner

Now, the output of inverting amplifier is given by $E_0(S)$.

$$\therefore \quad E_0(S) = -\left(\frac{R_1}{R_2}\right) E_1(S)$$

$$\text{or,} \quad \frac{E_0(S)}{E_1(S)} = -\frac{R_1}{R_2} \qquad (2)$$

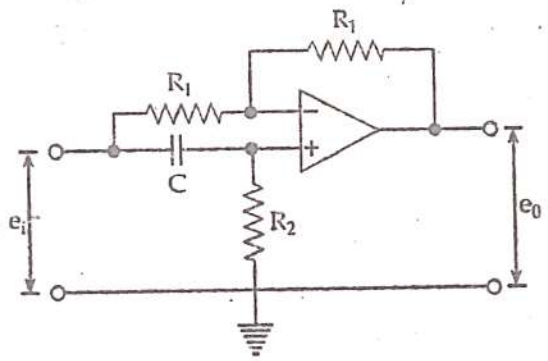
Multiplying equation (1) and (2); we get,

$$\frac{E_1(S)}{E_1(S)} \times \frac{E_0(S)}{E_1(S)} = -\left[1 + \frac{Z_2(S)}{Z_1(S)}\right]\left(\frac{R_1}{R_2}\right)$$

$$\text{or,} \quad \frac{E_0(S)}{E_1(S)} = -\frac{R_1}{R_2}\left(1 + \frac{\frac{R_4}{1+R_4CS}}{\frac{1+R_3CS}{CS}}\right) = -\frac{R_1}{R_2}\left[1 + \frac{R_4CS}{(1+R_4CS)(1+R_3CS)}\right]$$

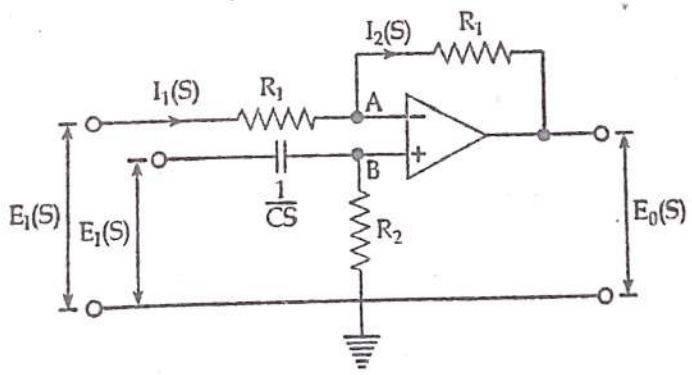
$$\therefore \quad \boxed{\frac{E_0(S)}{E_1(S)} = -\frac{R_1}{R_2}\left[\frac{(1+R_4CS)(1+R_3CS)+R_4CS}{(1+R_4CS)(1+R_3CS)}\right]}$$

4. Find the transfer function of given circuit. [2068 Chaitra]



Solution:

The circuit in S-domain becomes;



The voltage at point B is;

$$V_B = \left(\frac{R_2}{R_2 + \frac{1}{CS}}\right) E_1(S)$$

Since, the two inputs are virtually short; so that;

$$V_A = V_B = \left(\frac{R_2}{R_2 + \frac{1}{CS}}\right) E_1(S) \qquad (1)$$

For ideal op-Amp;

$$I_1(S) = I_2(S)$$

since, input impedance is infinite for ideal op-amp.

Scanned with CamScanner

or, $\frac{E_1(S) - V_A}{R_1} = \frac{V_A - E_0(S)}{R_1}$

or, $E_1(S) - V_A = V_A - E_0(S)$

or, $E_1(S) + E_0(S) = 2V_A$

Dividing both sides by $E_1(S)$; we get,

$$\frac{E_1(S) + E_0(S)}{E_1(S)} = \frac{2V_A}{E_1(S)}$$

or, $1 + \frac{E_0(S)}{E_1(S)} = \frac{2V_A}{E_1(S)}$ (2)

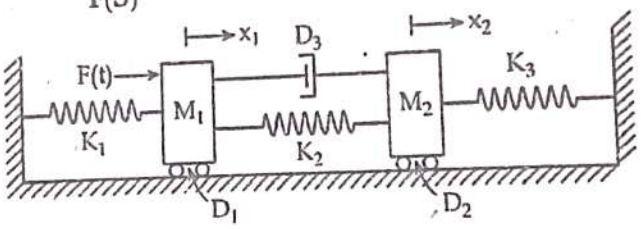
$\therefore \quad \frac{E_0(S)}{E_1(S)} = 2\frac{V_A}{E_1(S)} - 1$

Substituting the value of $\frac{V_A(S)}{E_1(S)}$ from the equation (1) in (2); we get,

$$\frac{E_0(S)}{E_1(S)} = 2\left(\frac{R_2}{R_2 + \frac{1}{CS}}\right) - 1 = \frac{2R_2CS}{1 + R_2CS} - 1 = \frac{2R_2CS - 1 - R_2CS}{1 + R_2CS}$$

$\therefore \quad \boxed{\frac{E_0(S)}{E_1(S)} = \frac{R_2CS - 1}{R_2CS + 1}}$; is the required transfer function.

5. Find the transfer function $\frac{x_2(S)}{F(S)}$, for a mechanical system of figure below. **[2071 Shrawan]**



Solution:

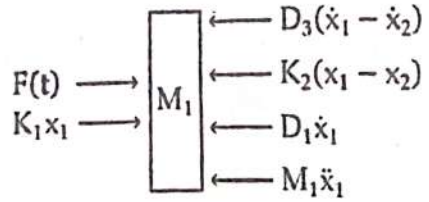
Here, we have to consider effect of friction on M_1 and M_2 since frictional constant (or frictional coefficient) D_1 and D_2 are given.

Frictional coefficient acts like a damping coefficient; so,

Frictional force = Frictional constant × Velocity

Now, reference direction is along the external force.

F.B.D. of mass M_1 is;



so, from D' Alembert's principle; we have,

$$F(t) = K_1x_1 + D_3(\dot{x}_1 - \dot{x}_2) + K_2(x_1 - x_2) + D_1\dot{x}_1 + M_1\ddot{x}_1$$

or, $F(t) = K_1x_1 + D_3\dot{x}_1 + K_2x_1 + D_1\dot{x}_1 + M_1\ddot{x}_1 - D_3\dot{x}_2 - K_2x_2$ (1)

Taking L.T. of equation (1); we have,

$$F(S) = [K_1 + K_2 + D_1S + D_3S + S^2M_1]x_1(S) - [K_2 + D_3S]x_2(S) \quad (2)$$

F.B.D. of mass M_2 is;

Scanned with CamScanner

$K_2(x_1 - x_2) \longrightarrow$ $D_3(\dot{x}_1 - \dot{x}_2) \longrightarrow$ $\boxed{M_2}$ $\longleftarrow K_3x_2$, $\longleftarrow M_2\ddot{x}_2$, $\longleftarrow D_2\dot{x}_2$

Again, from D' Alembert's principle; we have,

$$K_2(x_1 - x_2) + D_3(\dot{x}_1 - \dot{x}_2) = K_3x_2 + M_2\ddot{x}_2 + D_2\dot{x}_2$$

or, $$K_2x_1 - K_2x_2 + D_3\dot{x}_1 - D_3\dot{x}_2 = K_3x_2 + M_2\ddot{x}_2 + D_2\dot{x}_2$$

or, $$K_2x_1 + D_3\dot{x}_1 = K_2x_2 + K_3x_2 + D_2\dot{x}_2 + D_3\dot{x}_2 + M_2\ddot{x}_2 \quad (3)$$

Taking L.T. of equation (3); we have,

$$[K_2 + D_3S]x_1(S) = [K_2 + K_3 + D_2S + D_3S + S^2M_2]x_2(S)$$

$$\therefore \quad x_1(S) = \frac{(K_2 + K_3 + D_2S + D_3S + S^2M_2)}{(K_2 + D_3S)} \cdot x_2(S) \quad (4)$$

Substituting the value of equation (4) in equation (2); we get,

$$F(S) = [K_1 + K_2 + D_1S + D_3S + S^2M_1]x_1(S) - [K_2 + D_3S]x_2(S)$$

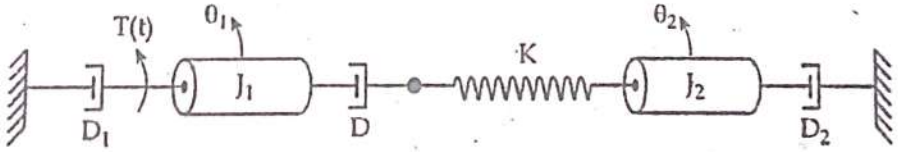
or, $$F(S) = (K_1 + K_2 + D_1S + D_3S + S^2M_1)\left[\frac{(K_2 + K_3 + D_2S + D_3S + S^2M_2)}{(K_2 + D_3S)}\right]x_2(S) - [K_2 + D_3S]x_2(S)$$

or, $$\frac{F(S)}{x_1(S)} = \frac{[K_1 + K_2 + D_1S + D_3S + S^2M_1][K_2 + K_3 + D_2S + D_3S + S^2M_2] - (K_2 + D_3S)^2}{(K_2 + D_3S)}$$

$$\therefore \quad \boxed{\frac{x_1(S)}{F(S)} = \frac{(K_2 + D_3S)}{[K_1 + K_2 + D_1S + D_3S + S^2M_1][K_2 + K_3 + D_2S + D_3S + S^2M_2] - (K_2 + D_3S)^2}} \quad (5)$$

Equation (5) is the required expression.

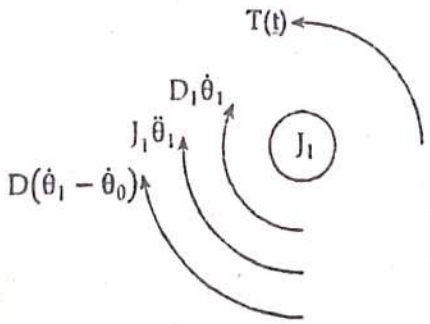
6. **Find transfer of system given below (Take angular displacement of J_2 as output). [2072 Kartik]**



Solution:

Here, two restoring torque D and K acting serially. So, we have to consider a mass-less mass have moment of inertia J_2 and angular displacement as Q_0.

F.B.D. of J_1;



so, from De' Alembert's principle; we have,

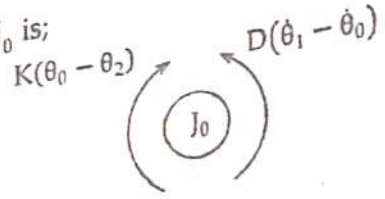
$$T(t) = D_1\dot{\theta}_1 + J_1\ddot{\theta}_1 + D(\dot{\theta}_1 - \dot{\theta}_0) = D_1\dot{\theta}_1 + J_1\ddot{\theta}_1 + D\dot{\theta}_1 - D\dot{\theta}_0 \quad (1)$$

Taking L.T. of equation (1); we have,

$$T(S) = (SD_1 + SD + S^2J)\theta_1(S) - SD\theta_0(S) \quad (2)$$

Scanned with CamScanner

Again, free body diagram (F.B.D.) of J_0 is;



From De' Alembert's principle; we have,

$$K(\theta_0 - \theta_2) = D(\dot{\theta}_1 - \dot{\theta}_0) \quad (3)$$

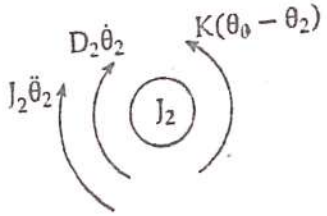
or, $K\theta_0 - K\theta_2 = D\dot{\theta}_1 - D\dot{\theta}_0$

or, $K\theta_0 + D\dot{\theta}_0 = K\theta_2 + D\dot{\theta}_1 \quad (4)$

Taking Laplace transform of equation (3); we get,

$$[K + SD]\theta_0(S) = K\theta_2(S) + SD\theta_1(S)$$

Again, F.B.D. of J_2 is;



From De' Alembert's principle; we have,

$$D_2\dot{\theta}_2 + J_2\ddot{\theta}_2 = K(\theta_0 - \theta_2) \quad (5)$$

or, $D_2\dot{\theta}_2 + J_2\ddot{\theta}_2 + K\theta_2 = K\theta_0$

Taking Laplace transform of equation (5); we get,

$$[K + SD_2 + S^2J_2]\theta_2(S) = K\theta_0(S)$$

$$\therefore \quad \theta_0(S) = \left(\frac{K + SD_2 + S^2J_2}{K}\right)\theta_2(S) \quad (6)$$

Now, from equation (4) and (6), substituting value of $\theta_0(S)$ in equation (4); we have,

$$\text{or,} \quad (K + SD)\left(\frac{K + SD_2 + S^2J_2}{K}\right)\theta_2(S) = K\theta_2(S) + SD\theta_1(S)$$

$$\text{or,} \quad (K + SD)(K + SD_2 + S^2J_2)\theta_2(S) = K^2\theta_2(S) + SDK\theta_1(S)$$

$$\text{or,} \quad \theta_2(S) = [(K + SD)(K + SD_2 + S^2J_2) - K^2] = SDK\theta_1(S) \quad (7)$$

Now, substituting the value of $\theta_1(S)$ from equation (7) to (2) and substituting the value of $\theta_0(S)$ from equation (6) to (2); we get,

$$T(S) = (SD_1 + SD + S^2J)\theta_1(S) - SD\theta_0(S)$$

$$\text{or,} \quad T(S) = (SD_1 + SD + S^2J)\left[\frac{(K + SD)(K + SD_2 + S^2J_2) - K^2}{SDK}\right]\theta_2(S) - SD\left(\frac{K + SD_2 + S^2J_2}{K}\right)\theta_2(S)$$

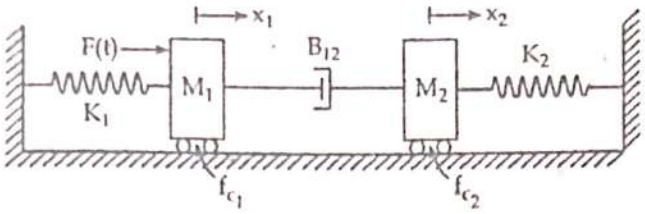
$$\text{or,} \quad T(S) = \frac{(SD_1 + SD + S^2J)[(K + SD)(K + SD_2 + S^2J_2) - K^2]\theta_2(S) - S^2D^2(K + SD_2 + S^2J_2)\theta_2(S)}{SDK}$$

$$\therefore \quad \boxed{\frac{\theta_2(S)}{T(S)} = \frac{SDK}{(SD_1 + SD + S^2J)[(K + SD)(K + SD_2 + S^2J_2) - K^2] - S^2D^2(K + SD_2 + S^2J_2)}}$$

which is the required transfer function.

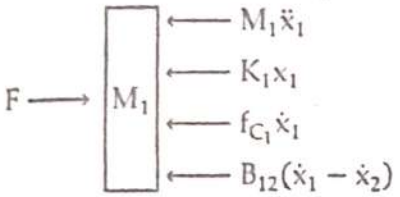
Scanned with CamScanner

7. **Construct the free body diagram and write the differential equations for the mechanical system given below and determine $\frac{x_2(S)}{F(S)}$. Also draw F–V and F–I analogy of the circuit below.** **[2073 Chaitra]**



Solution:

F.B.D for F(t) is;



From De-Alembert principle; we have,

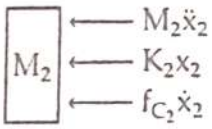
$$F(t) = M_1\left(\frac{d^2x_1}{\partial t^2}\right) + K_1x_1 + f_{C_1}\left(\frac{dx_1}{\partial t}\right) + B_{12}\left(\frac{d}{\partial t}\right)[x_1(t) - x_2(t)] \quad (1)$$

Taking Laplace transform; we have,

$$F(S) = S^2M_1X_1(S) + K_1X_1(S) + f_{C_1}X_1(S) + B_{12}S[x_1(S) - x_2(S)]$$

or, $$F(S) = [S^2M_1 + K_1 + Sf_{C_1} + SB_{12}]X_1(S) - SB_{12}X_2(S) \quad (2)$$

F.B.D. for M_2 is;



From De-Alembert principle; we have,

$$B_{12}\left(\frac{dx_1}{\partial t} - \frac{dx_2}{\partial t}\right) = M_2\left(\frac{d^2x_1}{\partial t^2}\right) + f_{C_2}\left(\frac{dx_2}{\partial t}\right) + K_2X_2(S) \quad (3)$$

Taking Laplace transform; we have,

$$SB_{12}[X_1(S) - X_2(S)] = S^2M_2X_2(S) + Sf_{C_2}X_2(S) + K_2X_2(S)$$

or, $$SB_{12}X_1(S) = [S^2M_2X_2(S) + S(f_{C_2} + B_{12})X_2(S) + K_2X_2(S)]$$

or, $$SB_{12}X_1(S) = [S^2M_2 + S(f_{C_2} + B_{12}) + K_2]X_2(S)$$

or, $$X_1(S) = \left[\frac{S^2M_2 + S(f_{C_2} + B_{12}) + K_2}{SB_{12}}\right]X_2(S) \quad (4)$$

Substituting the value of equation (4) in (2); we get,

$$F(S) = [S^2M_1 + S(f_{C_1} + B_{12}) + K_1] \times \left[\frac{S^2M_2 + S(f_{C_2} + B_{12}) + K_2}{SB_{12}}\right]X_2(S) - SB_{12}X_2(S)$$

or, $$\frac{F(S)}{x_2(S)} = \frac{[S^2M_1 + S(f_{C_1} + B_{12}) + K_1][S^2M_2 + S(f_{C_2} + B_{12}) + K_2] - S^2B_{12}^2}{SB_{12}}$$

$\therefore$ $$\frac{x_2(S)}{F(S)} = \frac{SB_{12}}{[S^2M_1 + S(f_{C_1} + B_{12}) + K_1][S^2M_2 + S(f_{C_2} + B_{12}) + K_2] - S^2B_{12}^2}$$

Scanned with CamScanner

F-V analogy of the circuit becomes;

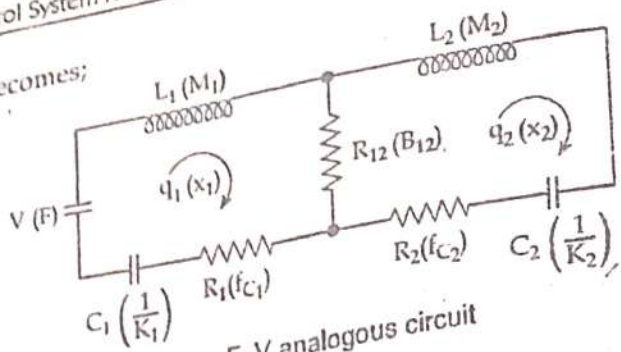


Figure: F–V analogous circuit

F-I analogy of the circuit becomes;

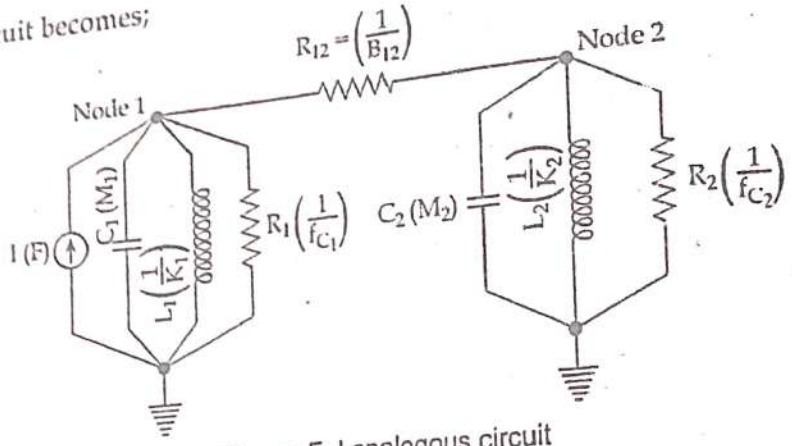
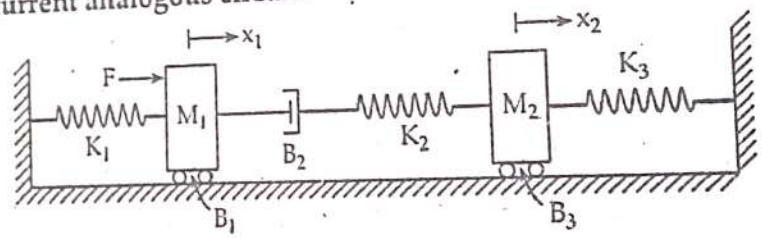


Figure: F–I analogous circuit

8. Find transfer function (consider displacement of mass M_2 as output for the given system. Al develop force-current analogous circuit. [2073 Shraw



Solution:

Here, B_2 and K_2 are in the series between two bodies M_1 and M_2. So, we have to consider effect of ma less mass M_0 and its displacement as x_3.

For the further steps

Refer the example 2.7

The F-I analogy of this circuit becomes;

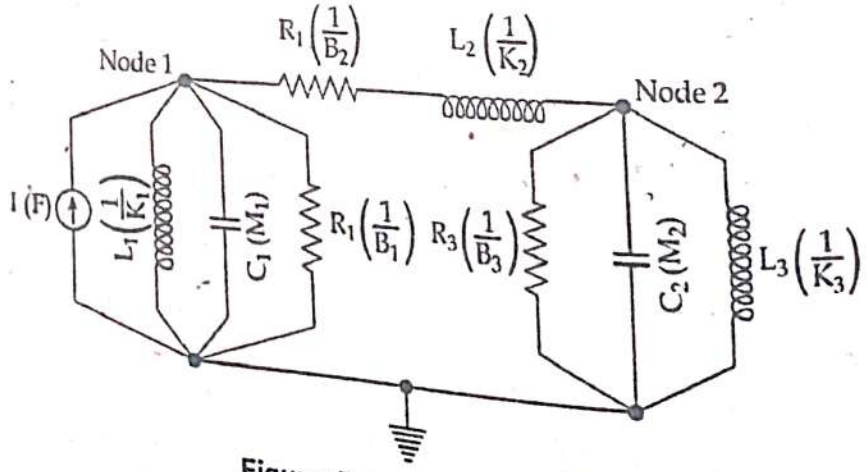
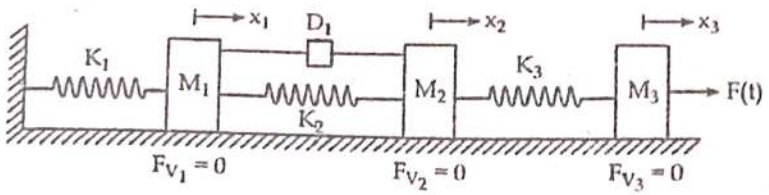


Figure: F–I analogous circuit

Scanned with CamScanner

9. **Find the transfer function, $\frac{x_2(S)}{F(S)}$, for the mechanical system of the figure below. Also draw F–V and F–I analogy circuit of the system.** **[2074 Ashwin]**



Solution:

Free body diagram of M_3 is;

$K_3(x_3 - x_2) \longleftarrow$ M_3 $\longrightarrow F(t)$

$M_3\ddot{x}_3 \longleftarrow$

According to De' Alembert's principle; we have,

$$F(t) = K_3(x_3 - x_2) + M_3\ddot{x}_3 = K_3x_3 + M_3\ddot{x}_3 - K_3x_2 \quad (1)$$

Taking Laplace transform of equation (1); we get,

$$F(S) = [S^2M_3 + K_3]X_3(S) - K_3X_2(S) \quad (2)$$

Free body diagram of M_2 is;

$D_1(\dot{x}_2 - \dot{x}_1) \longleftarrow$

$K_2(x_2 - x_1) \longleftarrow$ M_2 $\longrightarrow K_3(x_3 - x_2)$

$M_2\ddot{x}_2 \longleftarrow$

so, from De' Alembert's principle; we have,

$$K_3(x_3 - x_2) = D_1(\dot{x}_2 - \dot{x}_1) + K_2(x_2 - x_1) + M_2\ddot{x}_2$$

or, $K_3x_3 - K_3x_2 = D_1\dot{x}_2 - D_1\dot{x}_1 + K_2x_2 - K_2x_1 + M_2\ddot{x}_2$

or, $K_3x_3 = x_2(K_3 + K_2) + D_1\dot{x}_2 + M_2\ddot{x}_2 - D_1\dot{x}_1 - K_2x_1 \quad (3)$

Taking Laplace transform of equation (3); we get,

$$K_3x_3(S) = [K_2 + K_3 + S^2M_2 + SD_1] - [D_1S + K_2]X_1(S)$$

or, $$x_3(S) = \left(\frac{K_2 + K_3 + SD_1 + S^2M_2}{K_3}\right)X_2(S) - \left(\frac{SD_1 + K_2}{K_3}\right)X_1(S) \quad (4)$$

Now, free body diagram of M_1; we have,

$M_1\ddot{x}_1 \longleftarrow$

$K_1x_1 \longleftarrow$ M_2 $\longrightarrow D_1(\dot{x}_2 - \dot{x}_1)$

$\longrightarrow K_2(x_2 - x_1)$

From De' Alembert's principle; we have,

$$M_1\ddot{x}_1 + K_1x_1 = D_1(\dot{x}_2 - \dot{x}_1) + K_2(x_2 - x_1)$$

or, $M_1\ddot{x}_1 + K_1x_1 + D_1\dot{x}_1 + K_2x_1 = D_1\dot{x}_2 + K_2x_2 \quad (5)$

Taking Laplace transform of equation (5); we get,

$$[S^2M_1 + SD_1 + K_1 + K_2]X_1(S) = [SD_1 + K_2]X_2(S)$$

or, $$X_1(S) = \left(\frac{SD_1 + K_2}{S^2M_1 + SD_1 + K_1 + K_2}\right)X_2(S) \quad (6)$$

Scanned with CamScanner

From equation (4) and (6), substituting the value of $X_1(S)$; we get,

$$X_3(S) = \left(\frac{K_2 + K_3 + SD_1 + S^2M_2}{K_3}\right)X_2(S) - \left(\frac{SD_1 + K_2}{K_3}\right) \times \left(\frac{SD_1 + K_2}{S^2M_1 + SD_1 + K_1 + K_2}\right)X_2(S)$$

$$= \left[\frac{K_2 + K_3 + SD_1 + S^2M_2}{K_3} - \frac{(SD_1 + K_2)^2}{K_3(S^2M_1 + SD_1 + K_1 + K_2)}\right]X_2(S)$$

$$= \left[\frac{(K_2 + K_3 + SD_1 + S^2M_2)(K_1 + K_2 + S^2M_1 + SD_1) - (SD_1 + K_2)^2}{K_3(S^2M_1 + SD_1 + K_1 + K_2)}\right]X_2(S) \quad (7)$$

Now, substituting the value of equation (7) into (2); we get,

$$F(S) = (S^2M_3 + K_3)X_3(S) - K_3X_2(S)$$

$$= \left[\frac{(K_2 + K_3 + SD_1 + S^2M_2)(K_1 + K_2 + S^2M_1 + SD_1) - (SD_1 + K_2)^2}{K_3(S^2M_1 + SD_1 + K_1 + K_2)}\right](S^2M_3 + K_3)X_2(S) - K_3X_2(S)$$

$$= \frac{\left[(S^2M_3 + K_3)\{(K_2 + K_3 + SD_1 + S^2M_2)(K_1 + K_2 + S^2M_1 + SD_1) - (SD_1 + K_2)^2\} - K_3^2(S^2M_1 + SD_1 + K_1 + K_2)\right]}{K_3(S^2M_1 + SD_1 + K_1 + K_2)} \times X_2(S)$$

$$\therefore \quad \frac{X_2(S)}{F(S)} = \frac{K_3(S^2M_1 + SD_1 + K_1 + K_2)}{\left[(S^2M_3 + K_3)\{(K_2 + K_3 + SD_1 + S^2M_2)(K_1 + K_2 + S^2M_1 + SD_1) - (SD_1 + K_2)^2\} - K_3^2(S^2M_1 + SD_1 + K_1 + K_2)\right]}$$

Now, drawing F–V analogy; we get,

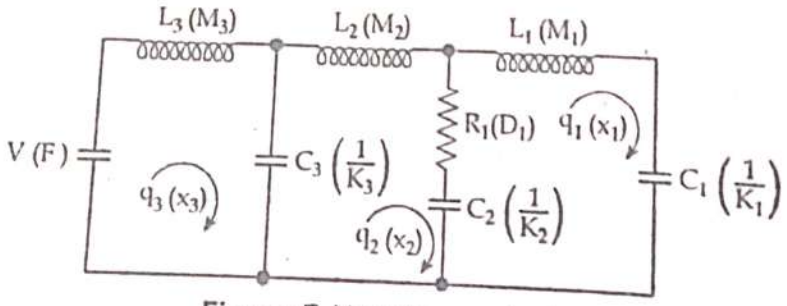


Figure: F–V analogous circuit

Now, drawing F–I analogy; we get,

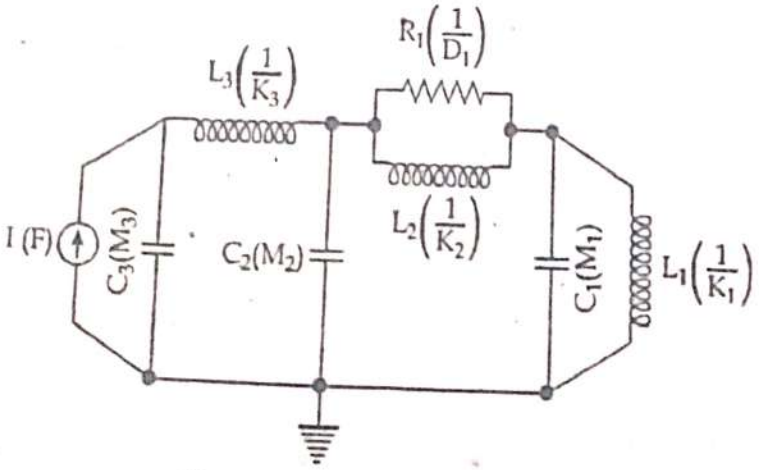
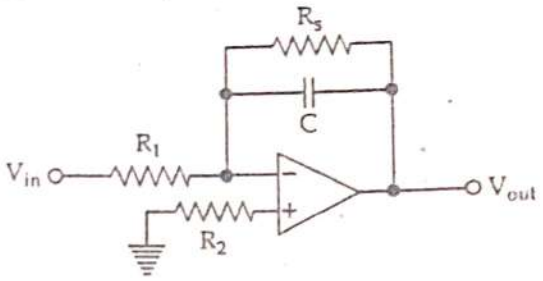


Figure: F–I analogous circuit

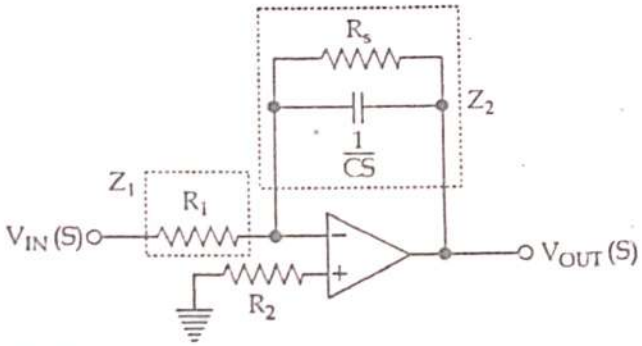
Scanned with CamScanner

10. Find transfer function of an OP-Amp model as below: [2074 Ashwin]



Solution:

The OP-Amp circuit is S-domain becomes;

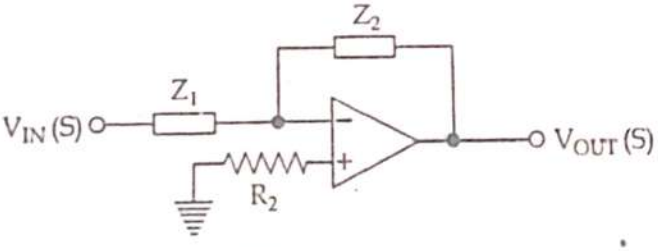


Here, resistor R_s is parallel with capacitor.

Let, $Z_1 = R_1$

and, $Z_2 = R_s \parallel \frac{1}{CS} = R_s \parallel \frac{R_s \times \frac{1}{CS}}{R_s + \frac{1}{CS}} = \frac{R_s}{1 + R_s CS}$

Now, the circuit becomes;



The input is fed into inverting terminal of an OP-Amp; so,

$$\frac{V_{out}(S)}{V_{in}(S)} = -\frac{Z_2}{Z_1} = -\frac{R_s}{R_1(1 + R_s CS)}$$

Here, input is given to negative terminal only, so no role of R_2 in transfer function.

Scanned with CamScanner

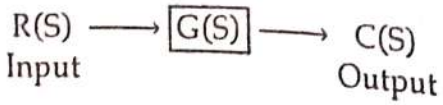
Chapter 3

SYSTEM TRANSFER FUNCTIONS AND RESPONSES

3.1 Combination of components to physical system 52
3.2 Block diagram algebra and system reduction 53
3.2.1 Rules for block diagram reduction 54
3.2.2 Problems regarding moving take–off point ahead and behind summing points and vice–versa 62
3.3 Signal flow graph: (SFG) method 65
3.3.1 Introduction of signal flow graph and drawback of block diagram reduction technique 65
3.3.2 Construction to draw S.F.G form block diagram 65
3.3.3 Mason's gain formula 66
3.4 Time response analysis 69
3.4.1 Types of test signal 70
3.4.2 Time response analysis of first order system 71
3.4.3 Time response analysis of second order system 74
3.4.4 Transient response characteristics/specifications 84
3.5 Effect of feedback on steady state gain, bandwidth, error magnitude, system dynamics or response, stability, and parameter variation 87
3.5.1 Effect of feedback on system dynamics or response 87
3.5.2 Effect of feedback on bandwidth 87
3.5.3 Effect of feedback on steady gain 88
3.5.4 effect of feedback on stability 88
3.5.5 Effect of parameter variations [Robustness] 88
3.5.6 Effect of feedback on error magnitude (Accuracy) 89
3.5.7 Effect of disturbances on control system 89

3.1 COMBINATION OF COMPONENTS TO PHYSICAL SYSTEM

Representation of single component or group of components by their transfer functions is known as block diagram representation.



Here,

Scanned with CamScanner

C(S) = Output

R(S) = Input

G(S) = Open loop transfer function

and, $\frac{C(S)}{R(S)}$ = Transfer function = G(S)

$\therefore$ $C(S) = G(S) \cdot R(S)$

A block may represent a single component or group of components, but each block is completely characterized by a transfer function. Transfer function is an expression which relates output to input in S-domain. The flow of system variables from one block to another is represented by the arrow. Also, the sum or differences of the signal are represented by the summing point. The application of one input source to two or more block is represented by take off point.

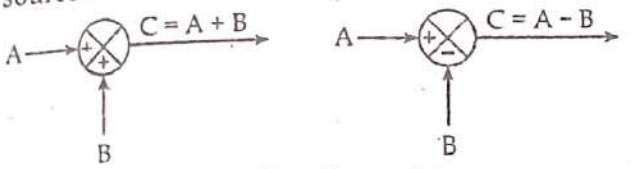


Figure: Summing point

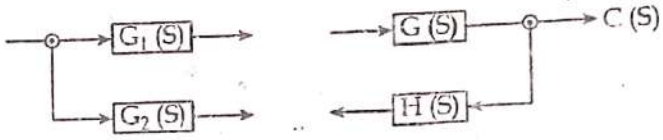
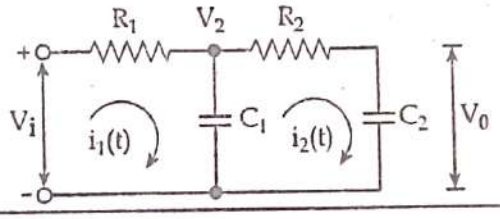


Figure: Take off points

3.2 BLOCK DIAGRAM ALGEBRA AND SYSTEM REDUCTION

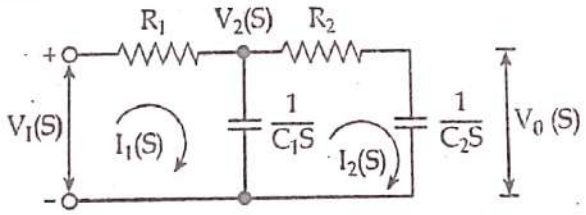
EXAMPLE 3.1

Draw the block diagram of circuit shown below. [2064 Falgun, 2067 Mangshir]



Solution:

The circuit in S-domain becomes;



Here,

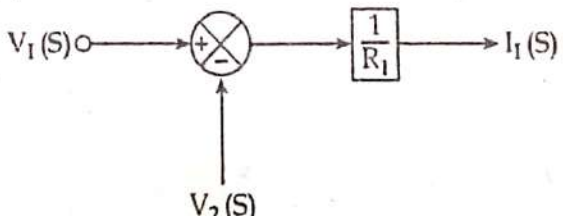
$$I_1(S) = \frac{V_I(S) - V_2(S)}{R_1} \quad (1)$$

$$V_2(S) = \frac{1}{C_1 S}[I_1(S) - I_2(S)] \quad (2)$$

$$I_2(S) = \frac{V_2(S) - V_0(S)}{R_2} \quad (3)$$

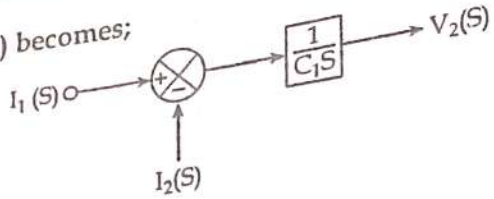
and, $V_0(S) = \frac{1}{C_2 S} I_2(S)$ (4)

The block diagram of equation (1) becomes;

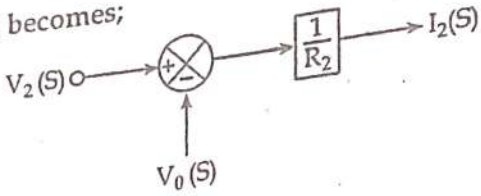


Scanned with CamScanner

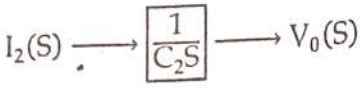
The block diagram of equation (2) becomes;



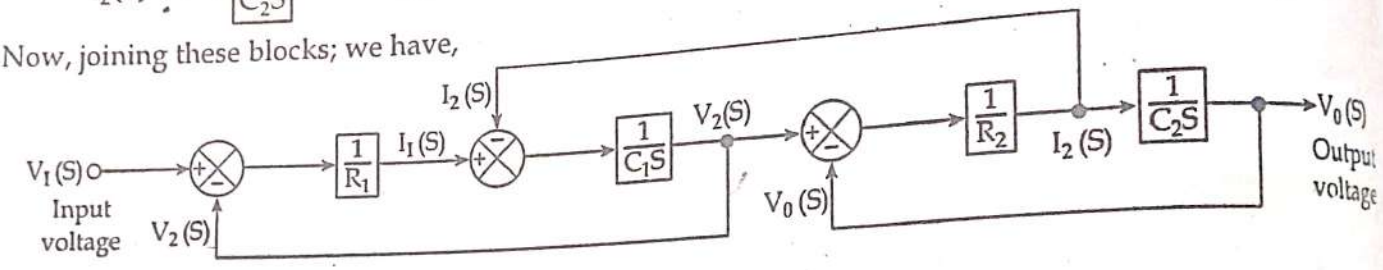
The block diagram of equation (3) becomes;



The block diagram of equation (4) becomes;

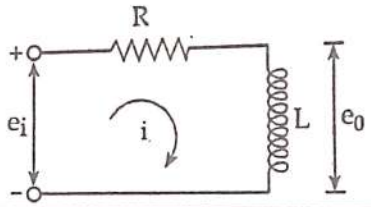


Now, joining these blocks; we have,



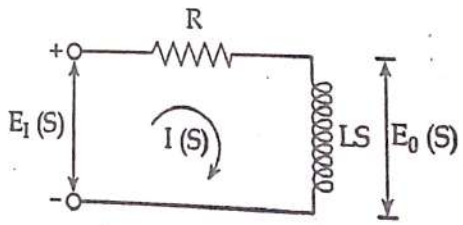
EXAMPLE 3.2

Draw the block diagram of circuit shown below.



Solution:

The circuit in S-domain becomes;

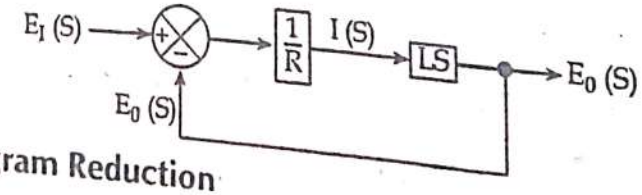


Here,

$$I(S) = \frac{E_I(S) - E_0(S)}{R}$$

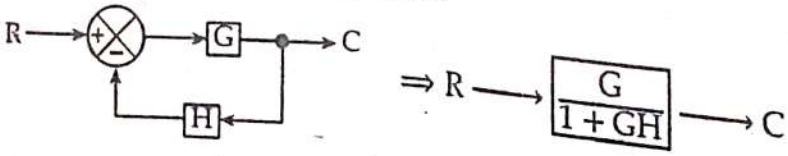
and, $E_0(S) = LS \times I(S)$

Now, drawing block diagram;



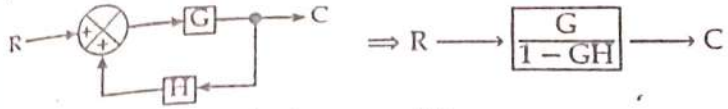
3.2.1 Rules for Block Diagram Reduction

Rule 1: Negative feedback circuit

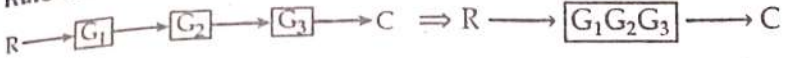


Scanned with CamScanner

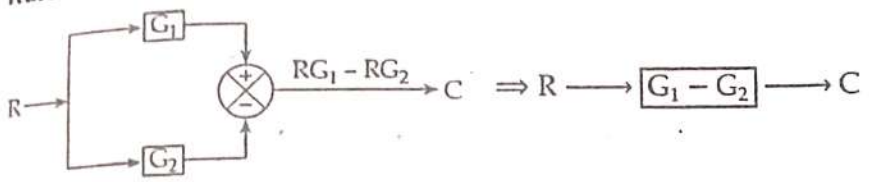
Rule 2: Positive feedback circuit

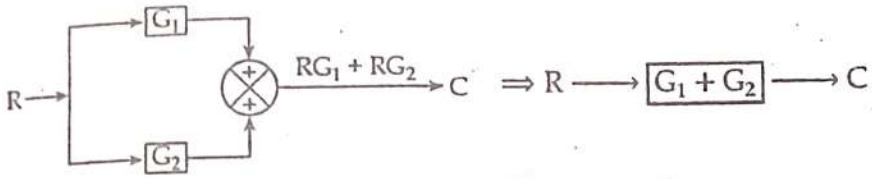


Rule 3: Blocks in series (or cascade)

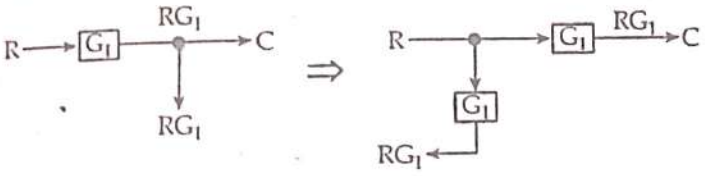


Rule 4: Blocks in parallel

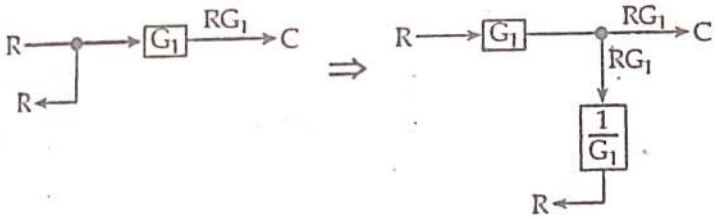




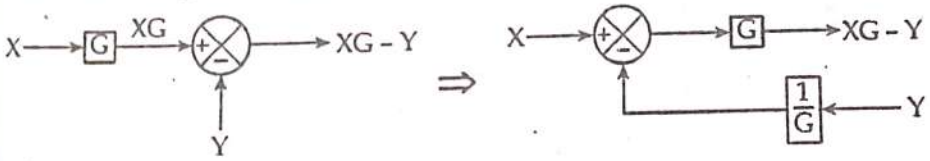
Rule 5: Moving take off point ahead of block



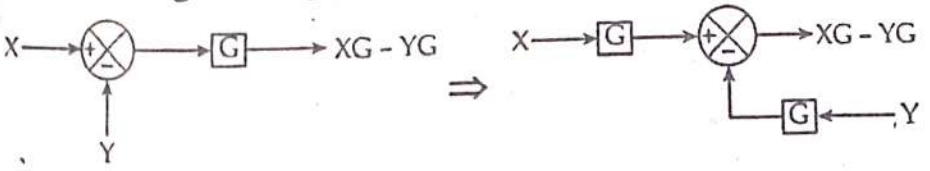
Rule 6: Moving take off point behind the block



Rule 7: Moving summing point ahead of block

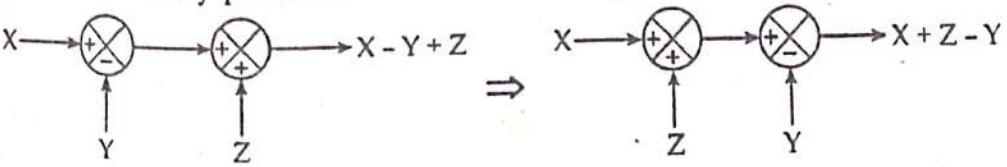


Rule 8: Moving summing point behind of block



Rule 9: Interchanging the summing points

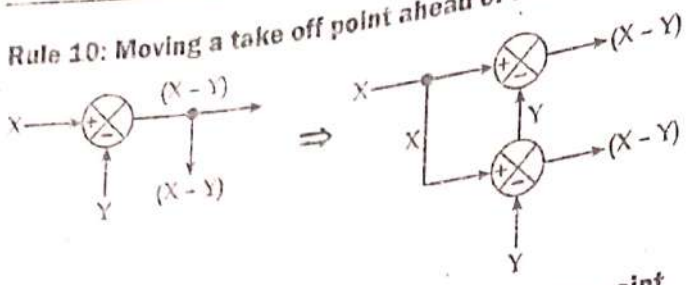
If there is no any point between two summing points



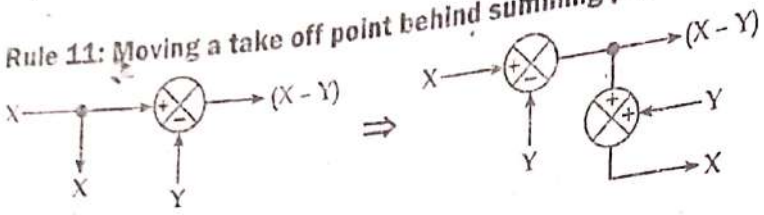
Take off point can also be exchanged.

Scanned with CamScanner

Rule 10: Moving a take off point ahead of summing points

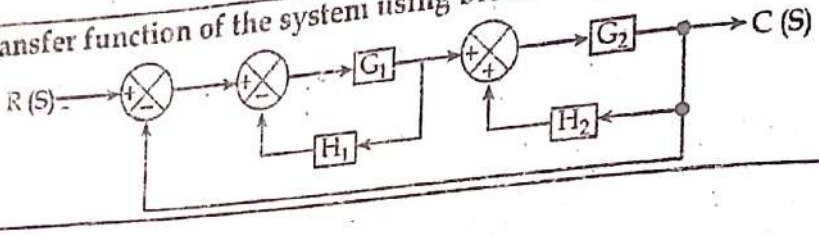


Rule 11: Moving a take off point behind summing point



EXAMPLE 3.3

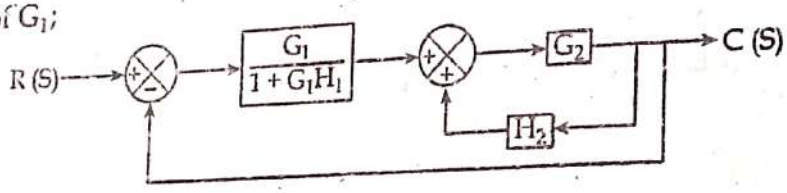
Derive the overall transfer function of the system using block diagram reduction technique.



Solution:

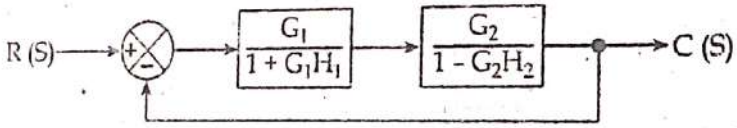
Step 1

Reducing inner loop of G_1;



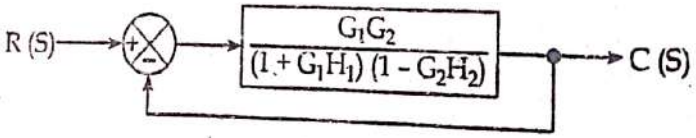
Step 2

Reducing inner loop of G_2;



Step 3

Blocks in cascade



Step 4

Here,

$$\text{Forward T.F.} = G(S) = \frac{G_1G_2}{(1+G_1H_1)(1-G_2H_2)}$$

$$\text{Backward T.F.} = H(S) = 1$$

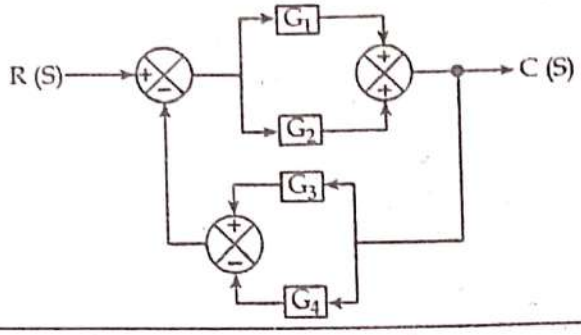
so, $$\text{T.F.} = \frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)}; \text{ for negative feedback}$$

$$\therefore \quad \frac{C(S)}{R(S)} = \frac{\frac{G_1G_2}{(1+G_1H_1)(1-G_2H_2)}}{1+\frac{G_1G_2}{(1+G_1H_1)(1-G_2H_2)}\cdot 1} = \frac{G_1G_2}{G_1G_2+(1+G_1H_1)(1-G_2H_2)}$$

Scanned with CamScanner

EXAMPLE 3.4

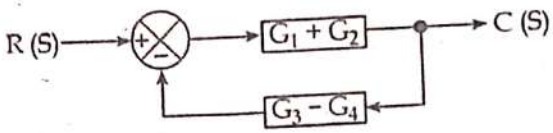
Determine the transfer function of given system using block diagram reduction technique.



Solution:

Step 1

Here, block G_1 and G_2 are in parallel similarly block G_3 and G_4 are also in parallel, reducing these loops; we get,



Step 2

$$G(S) = \text{Forward T.F.} = (G_1 + G_2)$$

$$\text{Backward T.F.} = H(S) = (G_3 - G_4)$$

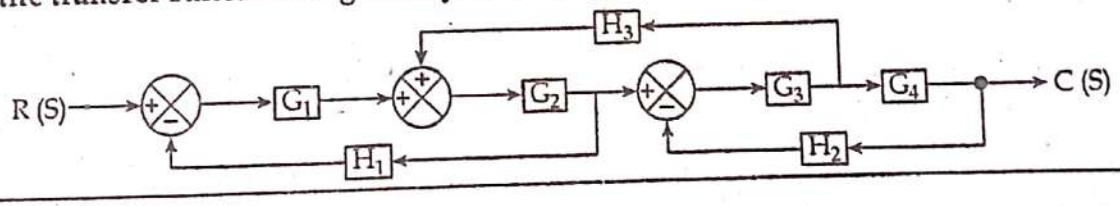
For negative feedback; we have,

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)H(S)}$$

$$\therefore \quad \boxed{\frac{C(S)}{R(S)} = \frac{G_1 + G_2}{1 + (G_1 + G_2)(G_3 - G_4)}}$$

EXAMPLE 3.5

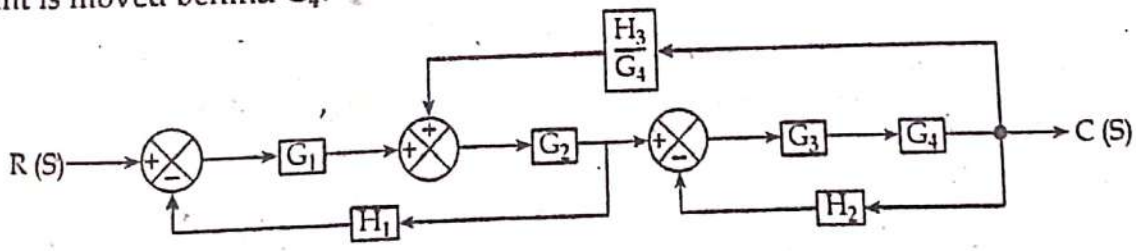
Determine the transfer function of given system using block diagram reduction technique.



Solution:

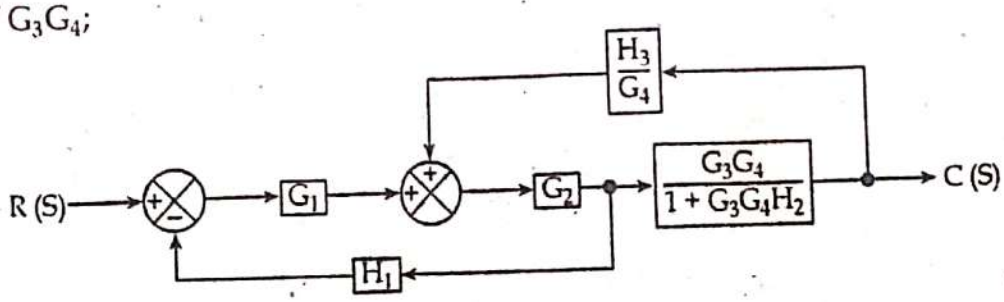
Step 1

Take off point is moved behind G_4.



Step 2

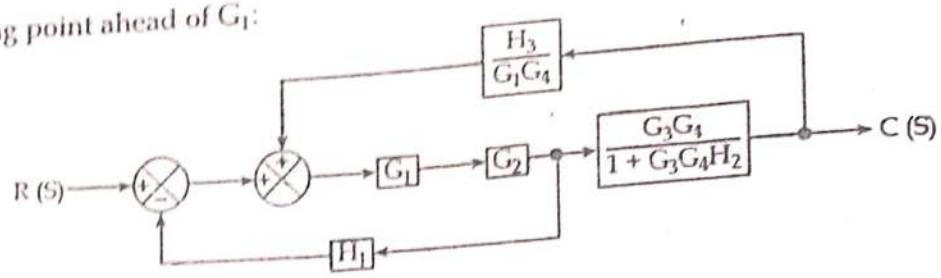
Reducing loop of G_3G_4;



Scanned with CamScanner

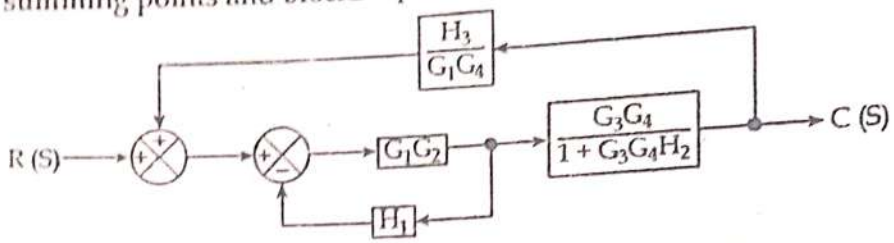
Step 3

Moving summing point ahead of G_1:



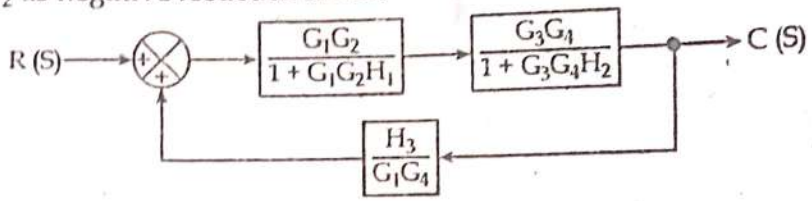
Step 4

Interchanging two summing points and blocks G_1 and G_2 are in cascade



Step 5

Reducing loop of G_1G_2 as negative feedback circuit



Step 6

Blocks in cascade and positive feedback circuit

$$G(S) = G_1(S) \times G_2(S) = \frac{G_1G_2G_3G_4}{(1+G_1G_2H_1)(1+G_3G_4H_2)}$$

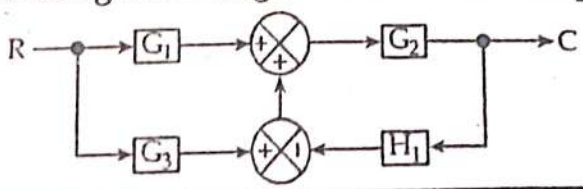
$$H(S) = \frac{H_3}{G_1G_4}$$

so, $$\text{T.F.} = \frac{C(S)}{R(S)} = \frac{G(S)}{1-G(S)H(S)} = \frac{\dfrac{G_1G_2G_3G_4}{(1+G_1G_2H_1)(1+G_3G_4H_2)}}{1-\dfrac{H_3}{G_1G_4}\times\dfrac{G_1G_2G_3G_4}{(1+G_1G_2H_1)(1+G_3G_4H_2)}}$$

$$\therefore \quad \boxed{\text{T.F.} = \frac{G_1G_2G_3G_4}{(1+G_1G_2H_1)(1+G_3G_4H_2)-G_2G_3H_3}}$$

EXAMPLE 3.6

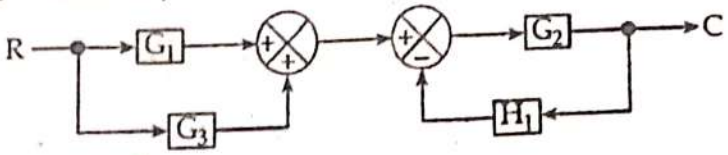
Determine the transfer function using block diagram reduction technique.



Solution:

Step 1

The block diagram can also written as;



Scanned with CamScanner

Step 2

Reducing loops;

i) G_1 and G_3 are parallel blocks

ii) H_1 as negative feedback

$$R \longrightarrow \boxed{G_1 + G_3} \longrightarrow \boxed{\frac{G_2}{1 + G_2H_1}} \longrightarrow C$$

Step 3

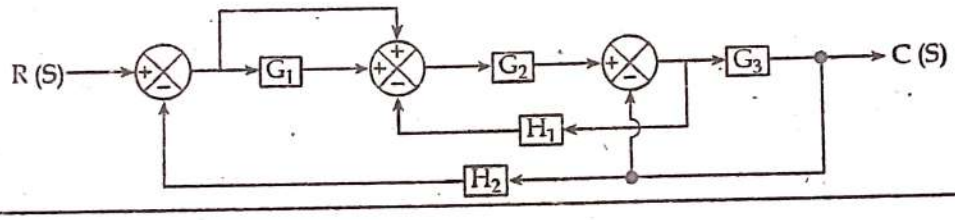
Blocks in cascade

$$R \longrightarrow \boxed{\frac{G_2(G_1 + G_3)}{1 + G_2H_1}} \longrightarrow C$$

$$\therefore \quad \text{Transfer function} = \frac{C}{R} = \frac{G_2(G_1 + G_3)}{1 + G_2H_1}$$

EXAMPLE 3.7

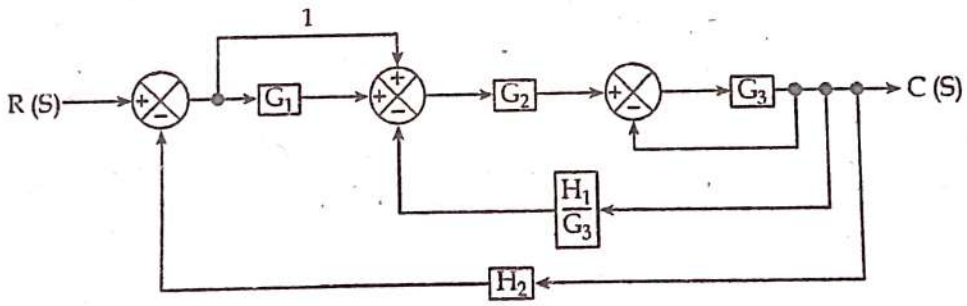
Reduce the block diagram and find out transfer function using block diagram reduction technique. **[2068 Baishakh]**



Solution:

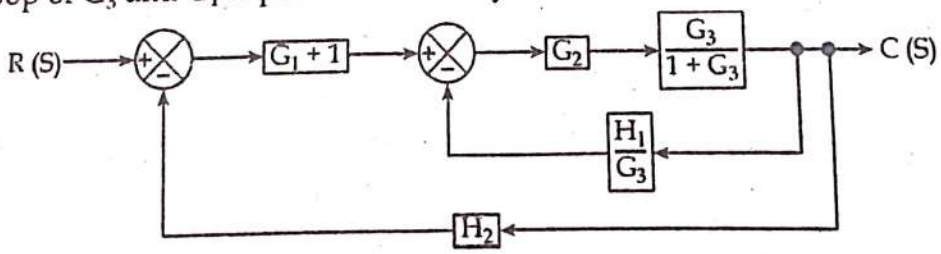
Step 1

Separating two feedback paths and moving take off point behind the block G_3.



Step 2

Reducing inner loop of G_3 and G_1 is parallel with unity.



Step 3

Reducing inner loops:

$$G(S) = G_1(S) \times G_2(S) = \frac{G_2G_3}{(1 + G_3)}$$

$$H(S) = \frac{H_1}{G_3}$$

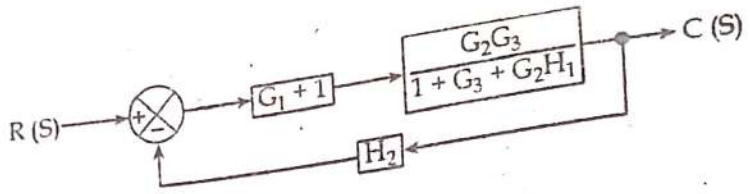
For negative feedback; we have,

Scanned with CamScanner

$$\text{T.F.} = \frac{G_2(S)}{1 + G(S)H(S)} = \frac{\frac{G_2G_3}{1+G_3}}{1 + \frac{G_2G_3}{1+G_3} \times \frac{H_1}{G_3}} = \frac{G_2G_3}{1 + G_3 + G_2H_1}$$

Step 4

The diagram becomes;



Step 5

$$G(S) = G_1(S) \times G_2(S) = \frac{(G_1 + 1)G_2G_3}{(1 + G_3 + G_2H_1)}$$

$$H(S) = H_2$$

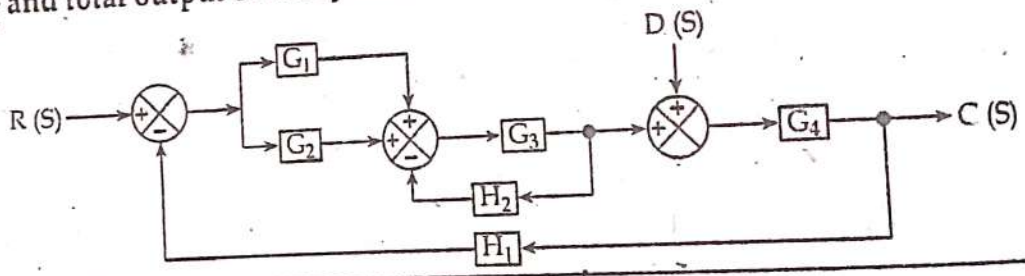
and, for reducing negative feedback; we have,

$$\text{T.F.} = \frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)H(S)} = \frac{\frac{(G_1 + 1)G_2G_3}{(1 + G_3 + G_2H_1)}}{1 + \left[\frac{(G_1 + 1)G_2G_3}{(1 + G_3 + G_2H_1)}\right] \times H_2}$$

∴ Transfer function $\boxed{(\text{T.F.}) = \frac{C(S)}{R(S)} = \frac{G_2G_3(1 + G_1)}{(1 + G_3 + G_2H_1) + (1 + G_1)G_2G_3H_2}}$

EXAMPLE 3.8

Find $\frac{C(S)}{R(S)}$, $\frac{C(S)}{D(S)}$ and total output of the system.



Solution:

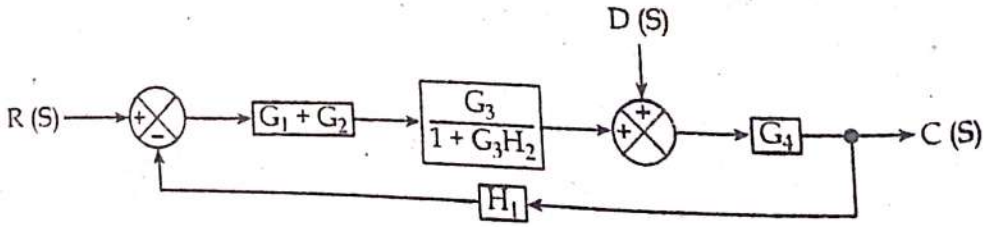
Here, two inputs R(S) and D(S);

Step 1

During block reduction;

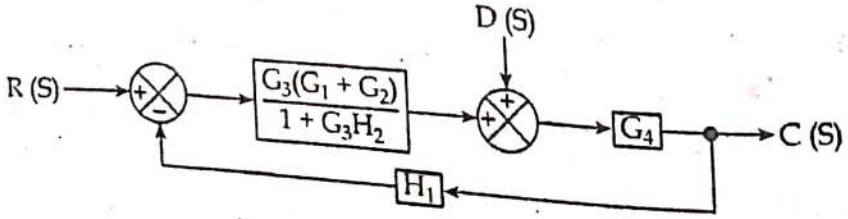
i) Block G_1 and G_2 are in parallel

ii) Reducing negative feedback circuit



Step 2

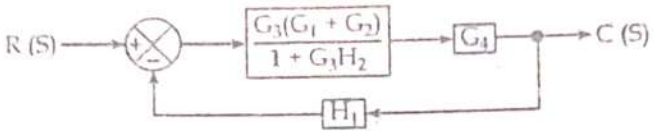
Blocks in cascade



Scanned with CamScanner

Step 3

To find $\frac{C(S)}{R(S)}$; let us assume $D(S) = 0$



Block in cascade;

Here,

$$G(S) = G_1(S) \times G_2(S) = \frac{G_3G_4(G_1 + G_2)}{(1 + G_3H_2)}$$

and, $H(S) = H_1$

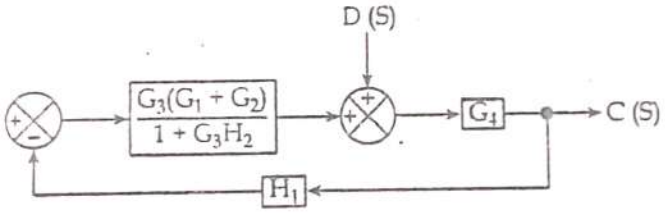
For negative feedback circuit; we have,

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)H(S)} = \frac{\frac{G_3G_4(G_1 + G_2)}{(1 + G_3H_2)}}{1 + \left[\frac{G_3G_4(G_1 + G_2)}{(1 + G_3H_2)}\right] \times H_1}$$

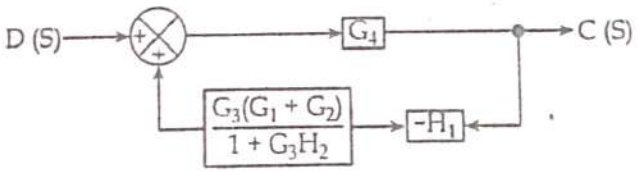
$$\therefore \quad \boxed{\frac{C(S)}{R(S)} = \frac{G_3G_4(G_1 + G_2)}{1 + G_3H_2 + G_3G_4(G_1 + G_2)H_1}}; \text{ for } D(S) = 0$$

Step 4

To find $\frac{C(S)}{D(S)}$; let us assume that $R(S) = 0$



i) Rearranging



ii) Reducing positive feedback circuit;

$$G(S) = G_4$$

and, $H(S) = H_1(S) \times H_2(S) = -\frac{G_3(G_1 + G_2)H_1}{(1 + G_3H_2)}$

so, Transfer function $= \frac{G(S)}{1 - G(S)H(S)} = \frac{G_4}{1 - \left[-\frac{G_3(G_1 + G_2)H_1}{(1 + G_3H_2)}\right] \times G_4}$

$$\therefore \quad \boxed{\frac{C(S)}{D(S)} = \frac{G_4(1 + G_3H_2)}{(1 + G_3H_2) + G_3G_4(G_1 + G_2)H_1}}; \text{ for } R(S) = 0$$

Step 5

Total output can be found out with the help of superposition theorem

Let, $C(S) = aC_1(S) + bC_2(S)$

where, a and b are constant.

Scanned with CamScanner

where, C(S) is the total output.

$C_1(S)$ is the output when $D(S) = 0$.

$C_2(S)$ is the o/p when $R(S) = 0$.

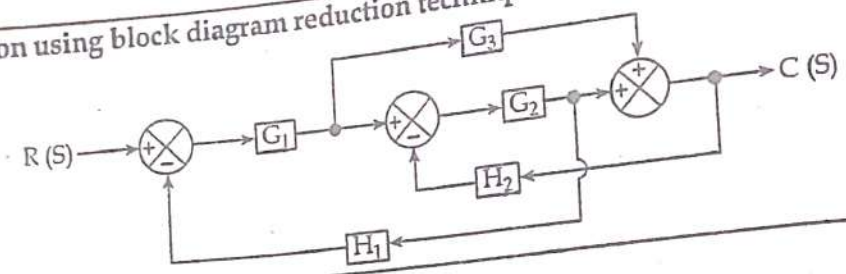
$$\therefore \quad C(S) = \frac{G_3G_4(G_1+G_2)}{1+G_3H_2+G_3G_4(G_1+G_2)H_1}R(S) + \frac{G_4(1+G_3H_2)}{(1+G_3H_2)+G_3G_4(G_1+G_2)H_1}D(S)$$

3.2.2 Problems Regarding Moving Take-Off Point Ahead and Behind Summing Points and Vice-Versa

Actually, we don't move take off point ahead or behind summing point and vice-versa. Actually, we suppose algebraic variables and reduced.

EXAMPLE 3.9

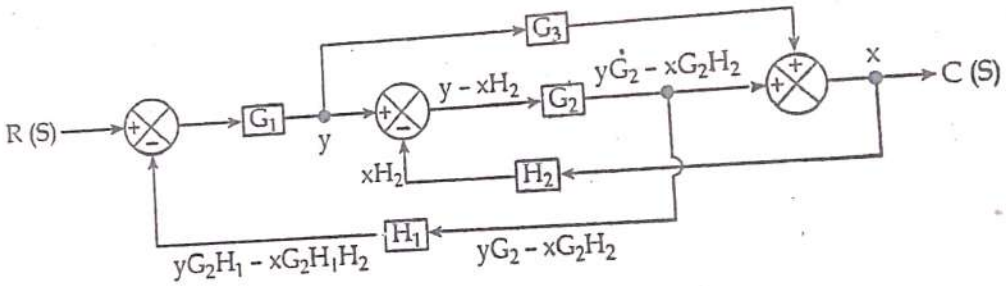
Find transfer function using block diagram reduction technique. [2067 Ashadh, 2068 Chaitra, 2070 Chaitra]



Solution:

Let us suppose 'x' at first take off point and y at last take off point.

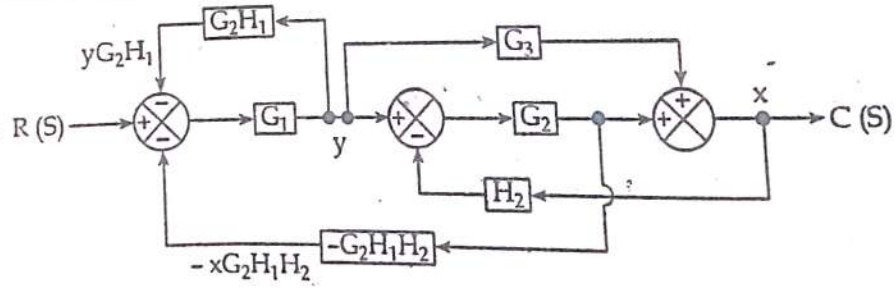
Step 1



Step 2

Now, splitting the H_1 feedback path;

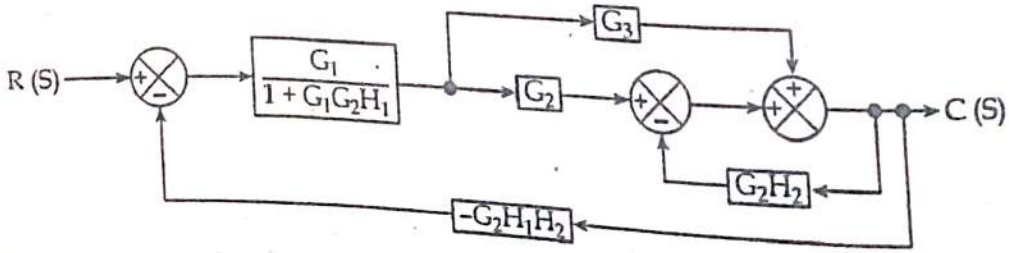
Take G_2H_1 from 'y' and $(-G_2H_1H_2)$ form point 'x'



Step 3

i) Now, reducing negative feedback circuit

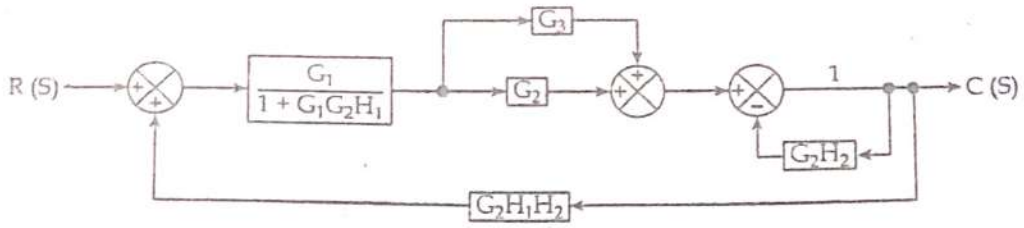
ii) Moving summing point behind G_2



Scanned with CamScanner

Step 4

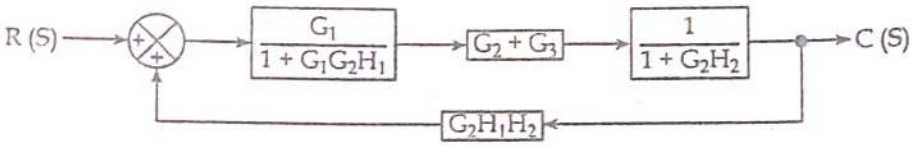
Interchanging summing points



Step 5

i) Reducing inner negative feedback loop of G_2H_2

ii) Blocks G_2 and G_3 are in parallel



Step 6

i) Blocks in cascade

$$G(S) = G_1(S) \times G_2(S) \times G_3(S) = \frac{G_1(G_2+G_3)}{(1+G_2H_2)(1+G_1G_2H_1)}$$

$$H(S) = G_2H_1H_2$$

ii) Reducing positive feedback circuit;

$$T.F. = \frac{C(S)}{R(S)} = \frac{G(S)}{1-G(S)H(S)}$$

$$= \frac{\frac{G_1(G_2+G_3)}{(1+G_2H_2)(1+G_1G_2H_1)}}{1-\frac{G_1G_2H_1H_2(G_2+G_3)}{(1+G_2H_2)(1+G_1G_2H_1)}}$$

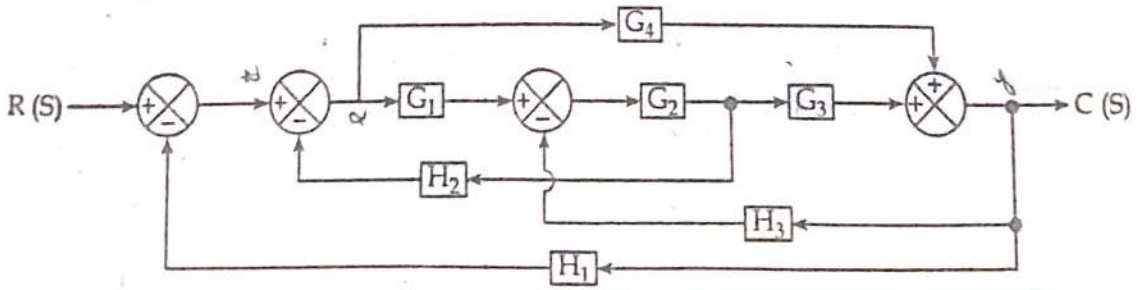
$$= \frac{G_1(G_2+G_3)}{(1+G_2H_2)(1+G_1G_2H_1)-G_1G_2(G_2+G_3)H_1H_2}$$

$$= \frac{G_1(G_2+G_3)}{1+G_1G_2H_1+G_2H_2+G_1G_2^2H_1H_2-G_1G_2^2H_1H_2-G_1G_2G_3H_1H_2}$$

$$\therefore \quad \boxed{T.F. = \frac{G_1(G_2+G_3)}{1+G_2H_2+G_1G_2H_1-G_1G_2G_3H_1H_2}}$$

EXAMPLE 3.10

Reduce the following block diagram model to obtain its overall transfer function. **[2070 Chaitra]**

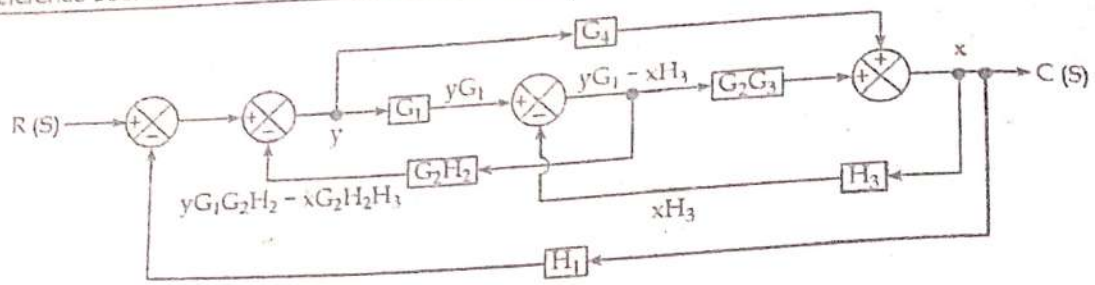


Solution:

Step 1

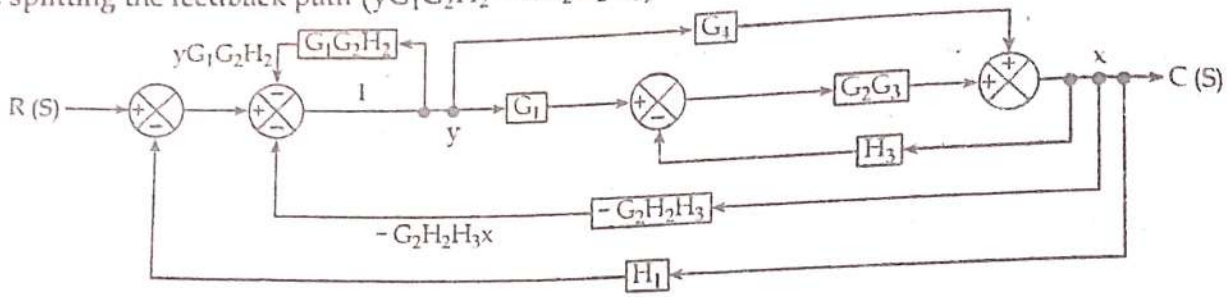
Let, 'x' is the first take off point and 'y' is the last take off point. Then, moving take off ahead of G_2

Scanned with CamScanner



Step 2

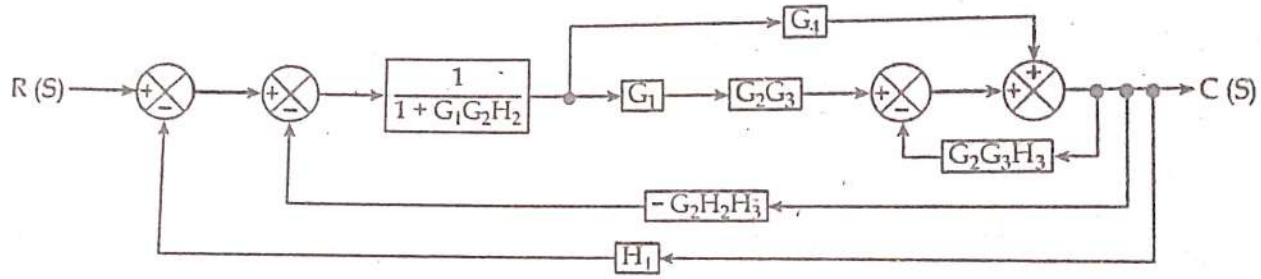
Now, splitting the feedback path $(yG_1G_2H_2 - xG_2H_2H_3)$



Step 3

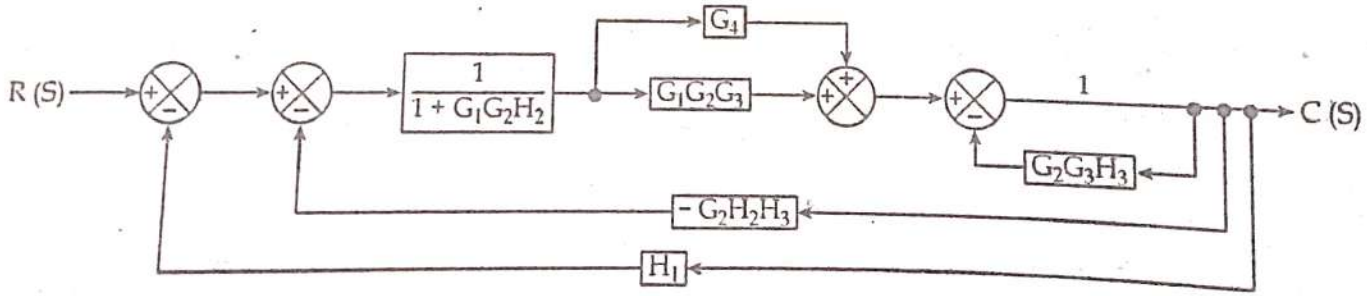
i) Reducing negative feedback path

ii) Moving summing point behind G_2G_3



Step 4

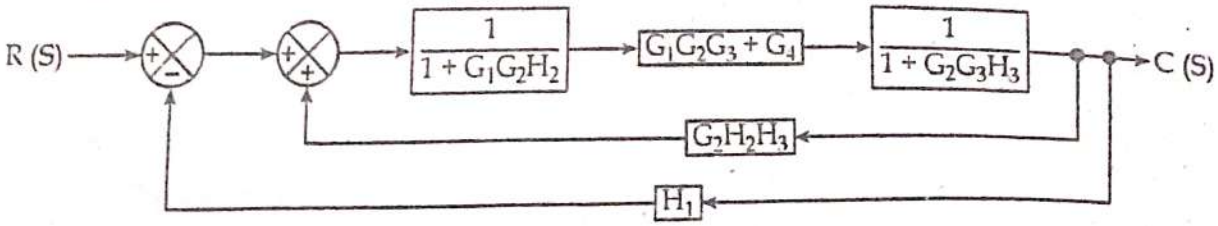
Interchanging summing points:



Step 5

i) Blocks in parallel

ii) Reducing negative feedback path



Step 6

i) Blocks in cascade

$$G(S) = G_1(S) \times G_2(S) \times G_3(S) = \frac{G_1G_2G_3 + G_4}{(1 + G_1G_2H_2)(1 + G_2G_3H_3)}$$

Scanned with CamScanner

$$H(S) = G_2H_2H_3$$

Then, transfer function is given by;

$$T.F. = \frac{G(S)}{1 - G(S)H(S)} = \frac{\dfrac{(G_1G_2G_3 + G_4)}{(1 + G_1G_2H_2)(1 + G_2G_3H_3)}}{1 - \dfrac{(G_1G_2G_3 + G_4)}{(1 + G_1G_2H_2)(1 + G_2G_3H_3)} \cdot G_2H_2H_3}$$

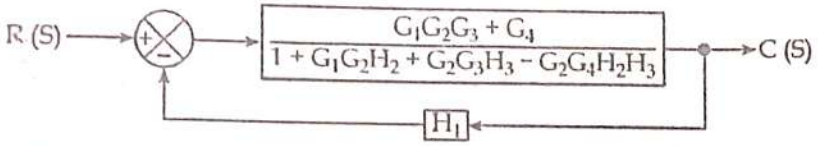
$$= \frac{G_1G_2G_3 + G_4}{(1 + G_1G_2H_2)(1 + G_2G_3H_3) - G_2H_2H_3(G_1G_2G_3 + G_4)}$$

$$= \frac{G_1G_2G_3 + G_4}{1 + G_1G_2H_2 + G_2G_3H_3 + G_1G_2^2G_3H_2H_3 - G_1G_2^2G_3H_2H_3 - G_2G_4H_2H_3}$$

$$\therefore \quad \boxed{T.F. = \frac{G_1G_2G_3 + G_4}{1 + G_1G_2H_2 + G_2G_3H_3 - G_2G_4H_2H_3}}$$

Step 7

The system now becomes;



Therefore, the required transfer function is;

$$T.F. = \frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)H(S)} = \frac{\dfrac{(G_1G_2G_3 + G_4)}{1 + G_1G_2H_2 + G_2G_3H_3 - G_2G_4H_2H_3}}{1 + \left(\dfrac{(G_1G_2G_3 + G_4)}{1 + G_1G_2H_2 + G_2G_3H_3 - G_2G_4H_2H_3}\right) \cdot G_2H_2H_3}$$

$$\therefore \quad \boxed{T.F. = \frac{G_1G_2G_3 + G_4}{1 + G_1G_2H_2 + G_2G_3H_3 - G_2G_4H_2H_3 + G_1G_2G_3H_1 + G_4H_1}}$$

3.3 SIGNAL FLOW GRAPH: (SFG) METHOD

3.3.1 Introduction of Signal Flow Graph and Drawback of Block Diagram Reduction Technique

Time consuming because in every step, a new block diagram has to be drawn.

- S.F.G method is a diagram which represents set of simultaneous equations and used to determine overall transfer functions of a control system.
- It is developed by S.J. Mason and applicable to linear system.

For example;

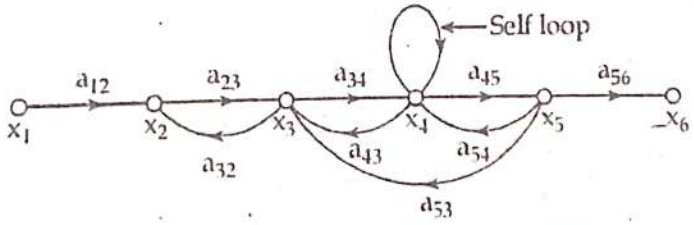


Figure: Example of signal flow graph (S.F.G.)

where, $a_{nn'}$ is the transfer function.

3.3.2 Construction to Draw S.F.G. From Block Diagram

i) All variables, summing points, and take off point are represented by nodes.

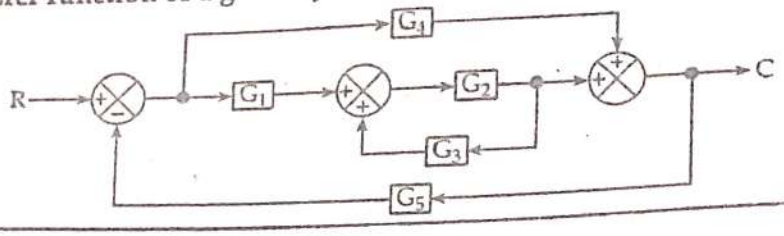
ii) If a summing point is placed before a take off point in the direction of signal flow, in such case represent the summing point and take off point by a signal node.

iii) If a summing point is placed after a take off point in the direction of signal flow, in such case, represent the summing point and take off point by separate nodes connected by a branch having transmittance unity.

Scanned with CamScanner

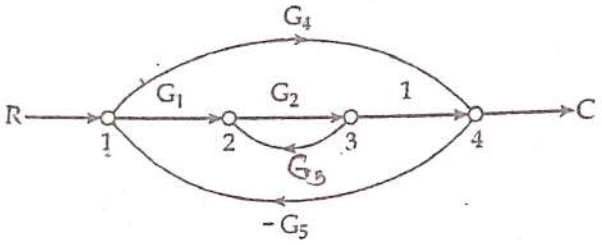
EXAMPLE 3.11

Find the overall transfer function of a given system using S.F.G method.



Solution:

Constructing S.F.G. of above system;



The transfer function can be calculated using masons gain formula;

3.3.3 Mason's Gain Formula

The overall transfer function or transmittance between input node and output node is given by;

$$T.F. = \frac{\sum g_k \Delta_k}{\Delta}$$ loop must be rotational

where, T.F. is the transfer function.

g_k is the gain of K^{th} forward path.

$\Delta = 1 - (\text{Sum of individual loop gain}) + (\text{Sum of all possible gain products of two non touching loops}) - (\text{Sum of all possible gain products of three non-touching loops}) + \ldots\ldots$

Δ_k = Part of Δ not touching K^{th} forward path

In the above question;

$g_1 = G_1G_2 \times 1 = G_1G_2$ $\quad [1 \to 2 \to 3 \to 4]$

$g_2 = G_4$ $\quad [1 \to 4]$

$L_1 = G_2G_3$ $\quad [2 \to 3 \to 2]$

$L_2 = G_1G_2 \times 1 \times (-G_5) = -G_1G_2G_5$ $\quad [1 \to 2 \to 3 \to 4 \to 1]$

$L_3 = G_4 \times (-G_5) = -G_4G_5$ $\quad [1 \to 4 \to 1]$

Now,

$$\Delta = 1 - [L_1 + L_2 + L_3] + [L_1L_3] - 0$$
$$= 1 - [G_2G_3 - G_1G_2G_5 - G_4G_5] + (G_2G_3)(-G_4G_5)$$
$$= 1 - G_2G_3 + G_1G_2G_5 + G_4G_5 - G_2G_3G_4G_5$$

Again,

Δ_1 = Part of Δ not touching $g_1 = 1 - 0 = 1$

Δ_2 = Part of Δ not touching $g_2 = 1 - L_1 = 1 - G_2G_3$

$$\text{Transfer function (T.F.)} = \frac{\sum g_k \Delta_k}{\Delta} = \frac{g_1\Delta_1 + g_2\Delta_2}{\Delta}$$

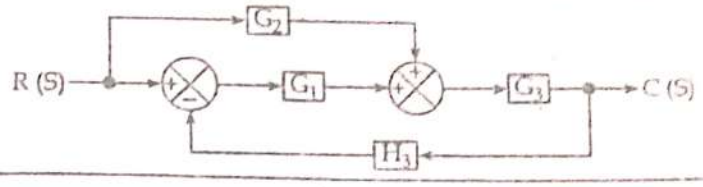
$$\therefore \quad \boxed{T.F. = \frac{G_1G_2 + G_4(1 - G_2G_3)}{1 - G_2G_3 + G_1G_2G_5 + G_4G_5 - G_2G_3G_4G_5}}$$

which is the required transfer function.

Scanned with CamScanner

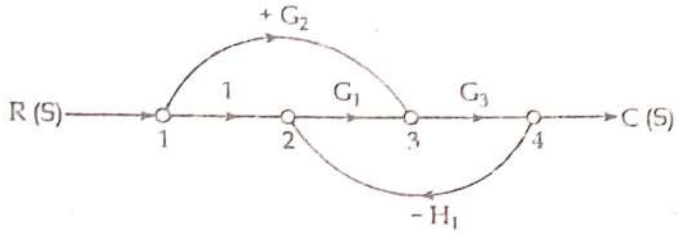
EXAMPLE 3.12

Draw the signal flow graph and find the transfer function of the given system.



Solution:

Drawing signal flow graph (SFG)



Here, forward paths are;

$g_1 = 1 \times G_1 \times G_3 = G_1G_3$ $[1 \to 2 \to 3 \to 4]$

$g_2 = G_2 \times G_3 = G_2G_3$ $[1 \to 3 \to 4]$

Individual loops;

$L_1 = G_1 \times G_3 \times (-H_1) = -G_1G_3H_1$ $[2 \to 3 \to 4 \to 2]$

and, $\Delta = 1 - [L_1] + 0 = 1 - L_1 = 1 + G_1G_3H_1$

Δ_1 = Part of Δ not touching $g_1 = 1 - 0 = 1$

Δ_2 = Part of Δ not touching $g_2 = 1 - 0 = 1$

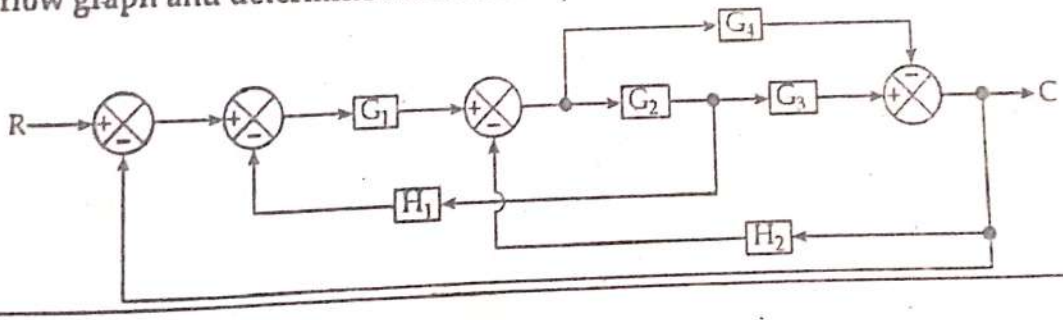
Now, using Mason gain formula;

$$\text{Transfer function (T.F.)} = \frac{C(S)}{R(S)} = \frac{\sum g_k \Delta_k}{\Delta} = \frac{g_1\Delta_1 + g_2\Delta_2}{\Delta} = \frac{G_1G_2 + G_2G_3}{1 + G_1G_3H_1}$$

$$\therefore \quad \boxed{\text{T.F.} = \frac{C(S)}{R(S)} = \frac{G_3(G_1 + G_2)}{1 + G_1G_3H_1}}$$

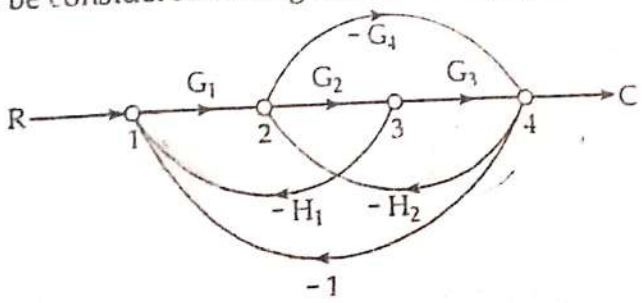
EXAMPLE 3.13

Draw signal flow graph and determine transfer function for the block diagram shown below.



Solution:

The two summing points can be considered as single since no any gain in between them.



Scanned with CamScanner

$[1 \to 2 \to 3 \to 4]$
$[1 \to 2 \to 4]$

Now, forward paths are;

$$g_1 = G_1 \times G_2 \times G_3 = G_1G_2G_3$$
$$g_2 = G_1 \times (-G_4) = -G_1G_4$$

$[1 \to 2 \to 3 \to 1]$
$[1 \to 2 \to 3 \to 4 \to 1]$
$[1 \to 2 \to 4 \to 1]$
$[2 \to 3 \to 4 \to 2]$
$[2 \to 4 \to 2]$

and, individual loops are;

$$L_1 = G_1 \times G_2 \times (-H_1) = -G_1G_2H_1$$
$$L_2 = G_1 \times G_2 \times G_3 \times (-1) = -G_1G_2G_3$$
$$L_3 = G_1 \times (-G_4) \times (-1) = G_1G_4$$
$$L_4 = G_2 \times G_3 \times (-H_2) = -G_2G_3H_2$$
$$L_5 = (-G_4) \times (-H_2) = G_4H_2$$
$$\Delta = 1 - [L_1 + L_2 + L_3 + L_4 + L_5] + [0]$$
$$= 1 - [G_1G_2H_1 - G_1G_2G_3 - G_1G_4 - G_2G_3H_2 + G_4H_2]$$
$$= 1 + G_1G_2H_1 + G_1G_2G_3 - G_1G_4 + G_2G_3H_2 - G_4H_2$$

Again,

Δ_1 = Part of Δ not touching $g_1 = 1 - 0 = 1$

Δ_2 = Part of Δ not touching $g_2 = 1 - 0 = 1$

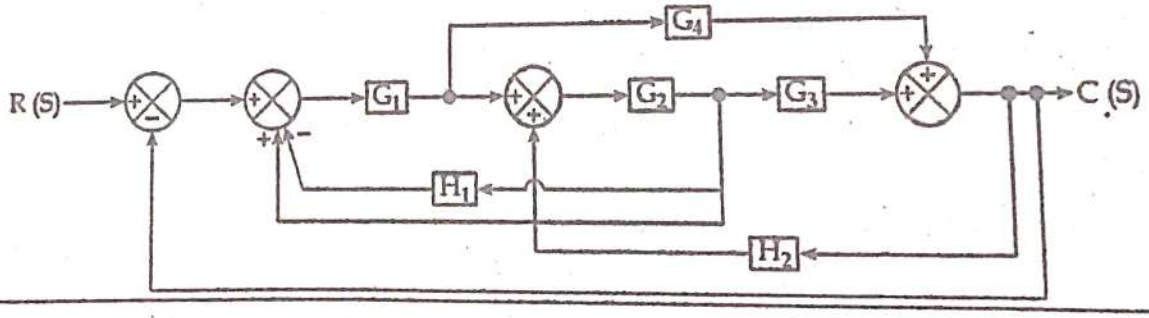
$$\text{Transfer function (T.F.)} = \frac{C}{R} = \frac{\sum g_k \Delta_k}{\Delta} = \frac{g_1\Delta_1 + g_2\Delta_2}{\Delta}$$

or, $$\frac{C}{R} = \frac{G_1G_2G_3 - G_1G_4}{1 + G_1G_2H_1 + G_1G_2G_3 - G_1G_4 + G_2G_3H_2 - G_4H_2}$$

$\therefore$ $$\boxed{\text{T.F.} = \frac{C}{R} = \frac{G_1(G_2G_3 - G_4)}{1 + G_1G_2H_1 + G_1G_2G_3 + G_2G_3H_2 - G_1G_4 - G_4H_2}}$$

EXAMPLE 3.14

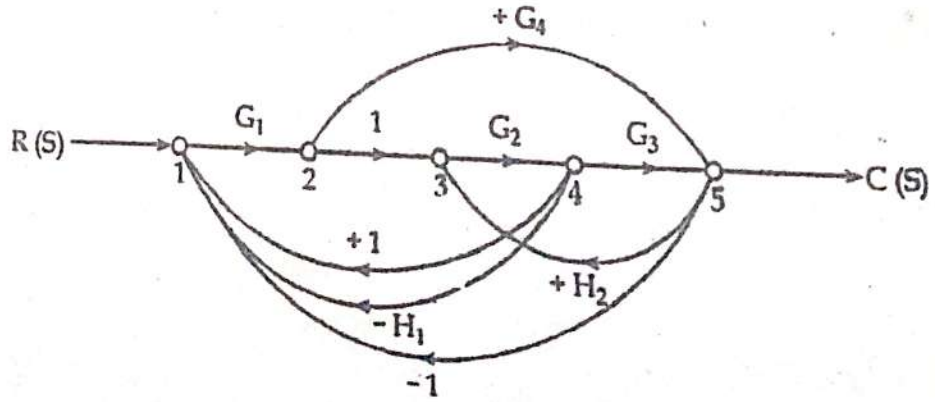
For the given block diagram draw the S.F.G. and find the ratio of $\frac{C(S)}{R(S)}$.



Solution:

Drawing signal flow graph (S.F.G.);

The front two summing points can be written as single; similarly, the last two take off points can be considered as single node.



Scanned with CamScanner

Forward paths;

$g_1 = G_1 \times 1 \times G_2 \times G_3 = G_1G_2G_3$ $[1 \to 2 \to 3 \to 4 \to 5]$

$g_2 = G_1 \times G_4 = G_1G_4$ $[1 \to 2 \to 5]$

Individual loops;

$L_1 = G_1 \times 1 \times G_2 \times 1 = G_1G_2$ $[1 \to 2 \to 3 \to 4 \to 1]$

$L_2 = G_2 \times G_3 \times H_2 = G_2G_3H_2$ $[3 \to 4 \to 5 \to 3]$

$L_3 = G_1 \times 1 \times G_2 \times (-H_1) = -G_1G_2H_1$ $[1 \to 2 \to 3 \to 4 \to 1]$

$L_4 = G_1 \times 1 \times G_2 \times G_3 \times (-1) = -G_1G_2G_3$ $[1 \to 2 \to 3 \to 4 \to 5 \to 1]$

$L_5 = G_1 \times G_4 \times (-1) = -G_1G_4$ $[1 \to 2 \to 5 \to 1]$

$L_6 = G_1 \times G_4 \times H_2 \times G_2 \times 1 = G_1G_2G_4H_2$ $[1 \to 2 \to 5 \to 3 \to 4 \to 1]$

$L_7 = G_1 \times G_4 \times H_2 \times G_2 \times (-H_1)$ $[1 \to 2 \to 5 \to 3 \to 4 \to 1]$

$= -G_1G_2G_4H_1H_2$

Now,

$$\Delta = 1 - [L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7] + 0$$
$$= 1 - [G_1G_2 + G_2G_3H_2 - G_1G_2H_1 - G_1G_2G_3 - G_1G_4 + G_1G_2G_4H_2 - G_1G_2G_4H_1H_2]$$
$$= 1 - G_1G_2 + G_1G_4 - G_2G_3H_2 + G_1G_2H_1 + G_1G_2G_3 - G_1G_2G_4H_2 + G_1G_2G_4H_1H_2$$

Again,

Δ_1 = Part of Δ not touching g_1 forward path $= 1 - 0 = 1$

Δ_2 = Part of Δ not touching g_2 forward path $= 1 - 0 = 1$

From Masons gain formula; we have,

$$\text{T.F.} = \frac{C(S)}{R(S)} = \frac{\sum g_k \Delta_k}{\Delta} = \frac{g_1\Delta_1 + g_2\Delta_2}{\Delta}$$

$$\therefore \quad \boxed{\text{T.F.} = \frac{G_1G_2G_3 + G_1G_4}{1 - G_1G_2 + G_1G_4 - G_2G_3H_2 + G_1G_2H_1 + G_1G_2G_3 - G_1G_2G_4H_2 + G_1G_2G_4H_1H_2}}$$

3.4 TIME RESPONSE ANALYSIS

- It indicates state of output of any system with change in time.
- Any system contains energy storing components like inductor, capacitor, mass, etc.
- If the energy states of energy storing (e.s.) elements is disturbed then it takes certain time to change from on state to another state.

Response

i) Transient response

ii) Steady state response

i) Transient response

The time required to change from one state to another state is known as transient time and the values of current and voltage during this period is known as transient response.

One state $\xrightarrow[\text{Transient time}]{}$ Another state

$\Downarrow$

Output

(Voltage, current, displacement, etc.)

ii) Steady state response

- Response of the system after transient state is terminated.
- It is stable state after transient state and should match to input of system otherwise steady state error is obtained.
- It is actually state of system at $t = \infty$.

Scanned with CamScanner

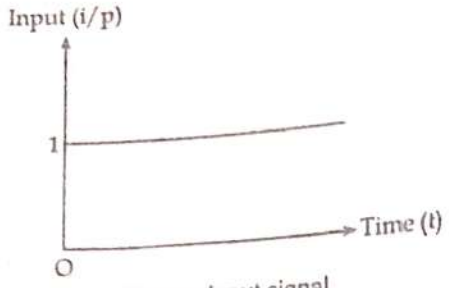


Figure: Input signal

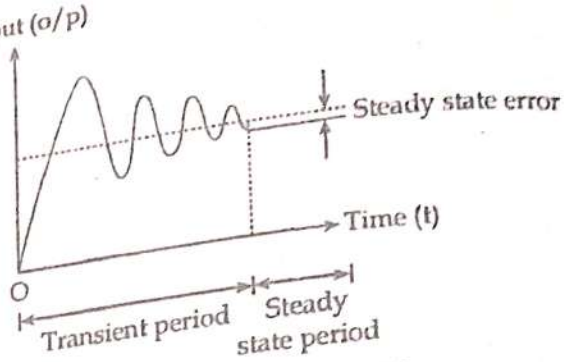
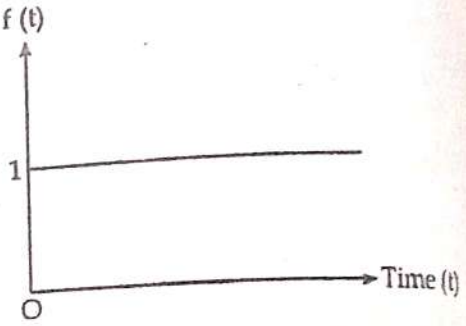


Figure: Output response

3.4.1 Types of Test Signal

i) Unit step function

$$f(t) = u(t) = 0, \quad t < 0$$
$$= 1, \quad t \geq 0$$



Laplace transform

$$u(s) = \int_0^\infty u(t)e^{-st}\,dt = \int_0^\infty 1 \cdot e^{-st}\,dt = \frac{1}{S}$$

Step function

$$u(t) = 0, \quad t < 0$$
$$= K, \quad t \geq 0$$

Step function is also known as **displacement function.** This input is used if system is subjected to sudden disturbances (changes) which are at constant nature.

ii) Impulse function

a) For shock inputs

$$\delta(t) = 0, \quad t \neq 0$$
$$= \infty, \quad t = 0$$

Mathematically,

$$\delta(t) = \frac{d}{dt}u(t)$$

Laplace transform

$$L[\delta(t)] = SL[u(t)] = S \times \frac{1}{S} = 1$$

- The pulse for which the duration tens to zero and amplitude tends to infinity is called as **impulse function.**
- Impulse function is also known as **delta function.**

This input is used if system is subjected to large disturbances for short time.

iii) Ramp function

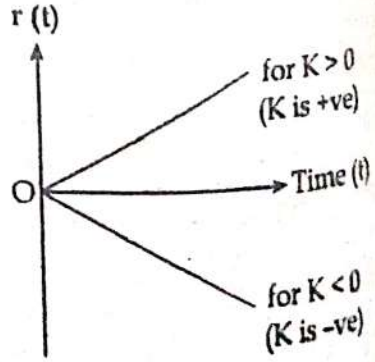
$$r(t) = 0, \quad t < 0$$
$$= Kt, \quad t \geq 0$$

For time varying signal

Laplace transform

$$L[r(t)] = \int_0^\infty Kt \cdot e^{-st}\,dt$$

$$\boxed{R(S) = \frac{K}{S^2}}$$



This input is used if system is subjected to gradual changes.

Scanned with CamScanner

Unit ramp

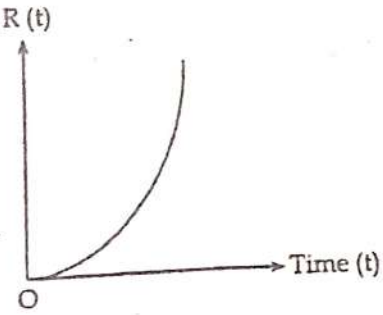
When K = 1; then,

$$\boxed{R(S) = \frac{1}{S^2}}$$

iv) **Parabolic signal**

$$R(t) = 0, \quad t < 0$$

$$= \frac{Kt^2}{2}, \quad t \geq 0$$



Laplace transform

$$L[R(t)] = \int_0^\infty \frac{Kt^2}{2} \cdot e^{-st}\, dt$$

$$\therefore \quad \boxed{R(S) = \frac{K}{S^3}}$$

This input is used if system is subjected to gradual change at a rate faster than ramp.

3.4.2 Time Response Analysis of First Order System

Physically this system represents RC, RL, thermal system etc. *i.e.*, one energy storing element and another elements which dissipates energy.

NOTE

Order of the system represents number of poles of the system.

Representation of first order system

Now,

$$\text{T.F.} = \frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)\,H(S)} = \frac{\frac{1}{ST}}{1 + \frac{1}{ST}} = \frac{1}{1 + ST}$$

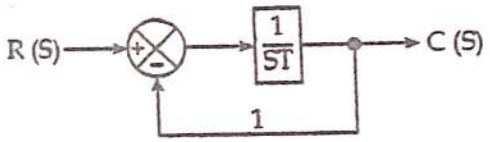


Figure: First order system with unity feedback

and, $H(S) = 1$

$$\therefore \quad R(S) \longrightarrow \boxed{\frac{1}{1 + ST}} \longrightarrow C(S)$$

Figure: First order system

i) **Response of first order system with unit step input function**

We know that;

Transfer function of the first order system;

$$\frac{C(S)}{R(S)} = \frac{1}{1 + ST}$$

For unit step input,

$$R(S) = \frac{1}{S}$$

$$\therefore \quad C(S) = \frac{1}{1 + ST} \times R(S) = \frac{1}{S(1 + ST)}$$

Using partial fraction; we have,

$$C(S) = \frac{A}{S} + \frac{B}{(1 + ST)}$$

$$\text{or,} \quad \frac{1}{S(1 + ST)} = \frac{A(1 + ST) + BS}{S(1 + ST)}$$

Scanned with CamScanner

or, $1 = A(1 + ST) + BS$

Put S = 0; we get,

$A = 1$

Put S = 1;

$1 = A(1 + T) + B$ [$\because A = 1$]

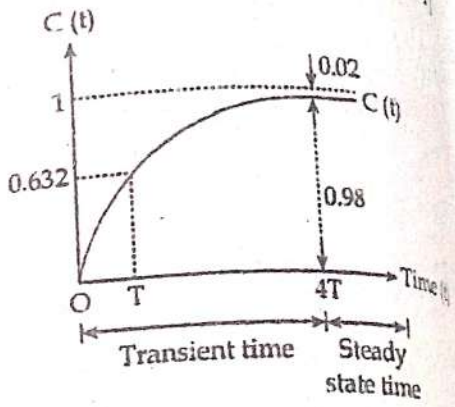
or, $1 = (1 + T) + B$

or, $B = -T$

so, $C(S) = \frac{1}{S} - \frac{T}{1+ST}$

or, $C(S) = \frac{1}{S} - \frac{T}{T\left(S + \frac{1}{T}\right)}$

or, $C(S) = \frac{1}{S} - \frac{1}{\left(S + \frac{1}{T}\right)}$



Taking inverse Laplace transform; we have,

$C(t) = 1 - e^{-(\frac{1}{T})t}$

- 'T' is the time constant of first order system

When t = T,

$C(t) = 0.632$

Hence, T is defined as time duration for the signal to attain 63.2% of steady state input value.

- It indicates how fast is the system response, if T is small the response is fast.

If t = 2T;

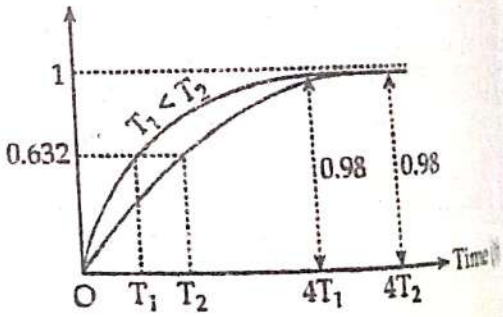
$C(t) = 0.864$

if t = 4T;

$C(t) = 0.982$

Settling time (4T)

Time required for any system to reach its response within 2% of desired output, represented by t_s.



$$\text{Error} = r(t) - C(t) = 1\left[1 - e^{-\frac{t}{\tau}}\right] = e^{-\frac{t}{\tau}}$$

For steady state error [e_{ss}]

$t = \infty$

∴ Error = 0

∴ $e_{ss} = 0$

ii) Response of first order system with unit ramp input

$$\frac{C(S)}{R(S)} = \frac{1}{1+ST}$$

For unit ramp;

$$R(S) = \frac{1}{S^2}$$

Then,

$$C(S) = \frac{1}{S^2(1+ST)}$$

Using partial fraction method;

Scanned with CamScanner

$$\frac{1}{S^2(1+ST)} = \frac{A}{S} + \frac{B}{S^2} + \frac{C}{1+ST}$$

Now,

$$B = \frac{1}{S^2(1+ST)} \times S^2\Big|_{S=0} = 1$$

Then, C(S) becomes;

$$\frac{1}{S^2(1+ST)} - \frac{1}{S^2} = \frac{A}{S} + \frac{C}{1+ST}$$

or, $$\frac{1-(1+ST)}{S^2(1+ST)} = \frac{A}{S} + \frac{C}{1+ST}$$

or, $$\frac{-ST}{S^2(1+ST)} = \frac{A}{S} + \frac{C}{1+ST}$$

or, $$\frac{-T}{S(1+ST)} = \frac{A}{S} + \frac{C}{1+ST}$$

Now,

$$A = \frac{-T}{S(1+ST)} \times S\Big|_{S=0} = \left(\frac{-T}{1}\right) = -T$$

and, $$C = \frac{-T}{S(1+ST)} \times (1+ST)\Big|_{S=-\frac{1}{T}} = \left(\frac{-T}{\frac{-1}{T}}\right) = T^2$$

Substituting; we get,

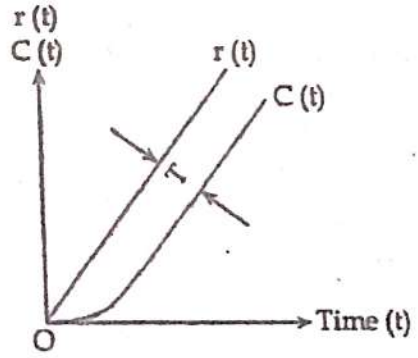
$$C(S) = \frac{-T}{S} + \frac{1}{S^2} + \frac{T^2}{1+ST} = \frac{1}{S^2} - \frac{T}{S} + \frac{T^2}{T\left(S+\frac{1}{T}\right)} = \frac{1}{S^2} - \frac{T}{S} + \frac{T}{\left(S+\frac{1}{T}\right)}$$

Taking inverse Laplace transform;

$$C(t) = t - T + Te^{-\frac{t}{T}}$$

$$\text{Error} = r(t) - C(t) = t - \left[t - T + Te^{-\frac{t}{T}}\right] = T - Te^{-\frac{t}{T}}$$

$$= T\left[1 - e^{-\frac{t}{T}}\right]$$



Steady state error;

$t = \infty$

$\therefore \quad e_{ss} = T$

From the figure, for smaller time constant, steady state error will be small and the speed of response will increase.

iii) **Response of first order system with unit impulse input**

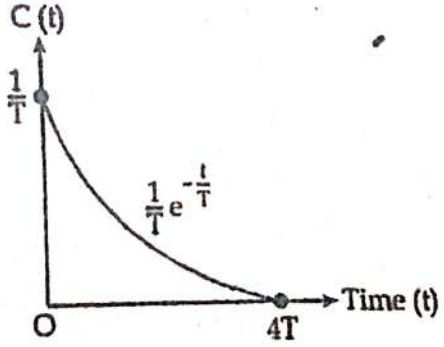
$$\frac{C(S)}{R(S)} = \frac{1}{1+ST}$$

For unit impulse input;

$$R(S) = 1$$

$\therefore \quad$ $$C(S) = \frac{1}{1+ST} = \frac{1}{T\left(S+\frac{1}{T}\right)}$$

Taking inverse Laplace transform;



Scanned with CamScanner

$$C(t) = \frac{1}{T} \cdot e^{-\frac{t}{T}}$$

Comparing;

For unit ramp; $C(t) = t - T + Te^{-\frac{t}{T}}$; $R(S) = \frac{1}{S^2}$

For unit step; $C(t) = 1 - e^{-\frac{t}{T}}$; $R(S) = \frac{1}{S}$ } Derivative

For unit impulse; $C(t) = \frac{1}{T} \cdot e^{-\frac{t}{T}}$; $R(S) = 1$ } Derivative

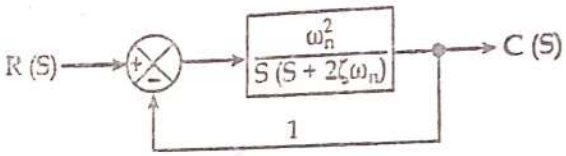
From above three equations, it is clear that the unit step input is the derivative of unit ramp input and unit impulse input is the derivative of unit step input.

3.4.3 Time Response Analysis of Second Order System

- System with two energy storing device.
  For example; Mass-spring system, LCR circuit, servo-system, etc.
- System with two poles
- The block diagram of second order control system with

$$G(S) = \frac{\omega_n^2}{S(S + 2\zeta\omega_n)}$$

and, $H(S) = 1$



Here,

$$G(S) = \frac{\omega_n^2}{S(S + 2\zeta\omega_n)}$$

and, $H(S) = 1$

$$\text{so, T.F.} = \frac{G(S)}{1 + G(S)H(S)} = \frac{\frac{\omega_n^2}{S(S + 2\zeta\omega_n)}}{1 + \frac{\omega_n^2}{S(S + 2\zeta\omega_n)}} = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

where, ω_n = Natural frequency of oscillation of second order system.

ζ = Damping ratio

$$= \frac{\text{Actual damping (damping coefficient or damping factor)}}{\text{Critical damping}} = \frac{\zeta\omega_n}{\omega_n}$$

$\zeta\omega_n$ = Damping factor (coefficient) or actual damping

i) Time response of second order system with unit step input

a) Undamped case ($\zeta = 0$)

Roots of characteristic equation are:

$$S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$$

or, $S^2 + 2 \times 0 \times \omega_n S + \omega_n^2 = 0$

or, $S^2 + \omega_n^2 = 0$

i.e., roots are equal and complex.

Therefore, the system is oscillating to its natural frequency ω_n.

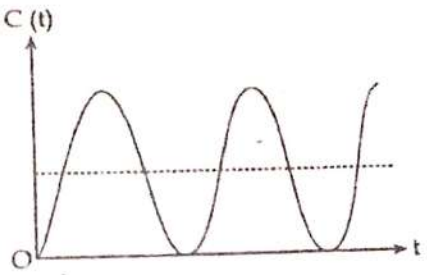
Scanned with CamScanner

so, $\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + \omega_n^2}$

For unit step input,

$R(S) = \frac{1}{S}$

so, $C(S) = \frac{\omega_n^2}{S(S^2 + \omega_n^2)} = \frac{1}{S} - \frac{S}{S^2 + \omega_n^2}$



Taking inverse Laplace transform;

$C(t) = 1 - \cos \omega_n t$ $\left[\because L(\cos \omega t) = \frac{S}{S^2 + \omega^2}\right]$

i.e., oscillating part remains forever.

b) **Critically damped ($\zeta = 1$) case**

Roots of characteristics equation are:

$S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$

or, $S^2 + 2\omega_n S + \omega_n^2 = 0$

or, $(S + \omega_n)^2 = 0$

Roots are real and equal.

System returns to equilibrium as quickly as possible without oscillating.

so, $\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} = \frac{\omega_n^2}{(S + \omega_n)^2}$

For unit step input,

$R(S) = \frac{1}{S}$

so, $C(S) = \frac{\omega_n^2}{(S + \omega_n)^2} = \frac{A}{S} + \frac{B}{S + \omega_n} + \frac{C}{(S + \omega_n)^2}$

Now,

$$A = \frac{\omega_n^2}{(S + \omega_n)^2} \times S\Big|_{S=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$C = \frac{\omega_n^2}{S(S + \omega_n)^2} \times (S + \omega_n)^2\Big|_{S=-\omega_n} = \frac{\omega_n^2}{(-\omega_n)} = -\omega_n$$

Now,

$$\frac{\omega_n^2}{S(S + \omega_n)^2} = \frac{1}{S} - \frac{\omega_n}{(S + \omega_n)^2} + \frac{B}{(S + \omega_n)}$$

or, $$\frac{\omega_n^2}{S(S + \omega_n)^2} - \frac{1}{S} + \frac{\omega_n}{(S + \omega_n)^2} = \frac{B}{(S + \omega_n)}$$

or, $$\frac{\omega_n^2 - (S + \omega_n)^2 + \omega_n S}{S(S + \omega_n)^2} = \frac{B}{(S + \omega_n)}$$

or, $$\frac{\omega_n^2 + \omega_n S - S^2 - 2S\omega_n - \omega_n^2}{S(S + \omega_n)^2} = \frac{B}{(S + \omega_n)}$$

or, $$\frac{-S^2 - \omega_n S}{S(S + \omega_n)} = B$$

Scanned with CamScanner

or, $B = \frac{-S(S+\omega_n)}{S(S+\omega_n)} = 1$

Substituting values of A, B and C; we get,

$$C(S) = \frac{1}{S} - \frac{1}{S+\omega_n} + \frac{\omega_n}{(S+\omega_n)^2}$$

Taking inverse Laplace transform; we have,

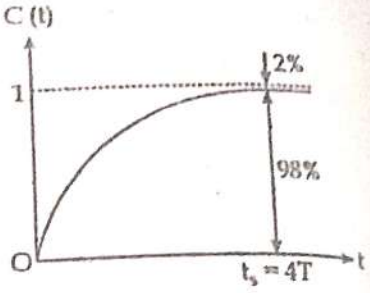
$$C(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$

or, $C(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$

From the graph; we have,

Time constant of the system $(T) = \frac{1}{\omega_n}$

Settling time $(t_s) = 4T = \frac{4}{\omega_n}$



At value of critical damping, the oscillation just disappeared.

c) Over-damped case ($\zeta > 1$)

Roots of characteristics equation are:

$$S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$$

or, $(S + \zeta\omega_n)^2 - \zeta^2\omega_n^2 + \omega_n^2 = 0$

or, $(S + \zeta\omega_n)^2 - \omega_n^2(\zeta^2 - 1) = 0$

Roots are real and unequal.

System returns (exponentially decays) to equilibrium without oscillating.

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} = \frac{\omega_n^2}{(S+\zeta\omega_n)^2 - \omega_n^2(\zeta^2-1)}$$

For unit step i/p;

$$R(S) = \frac{1}{S}$$

Then,

$$C(S) = \frac{\omega_n^2}{S\left[(S+\zeta\omega_n)^2 - \omega_n^2(\zeta^2-1)\right]}$$

By partial fraction;

$$C(S) = \frac{\omega_n^2}{S\left[(S+\zeta\omega_n)^2 - \omega_n^2(\zeta^2-1)\right]} = \frac{A}{S} + \frac{B}{(S+\zeta\omega_n+\omega_d)} + \frac{C}{(S+\zeta\omega_n-\omega_d)}$$

where, $\omega_d = \omega_n\sqrt{\zeta^2-1}$ = Damped frequency of oscillation.

ω_n = Damped frequency of oscillation.

$$A = \frac{\omega_n^2}{S\left[(S+\zeta\omega_n)^2 - \omega_d^2\right]} \times (S)\Big|_{S=0} = \frac{\omega_n^2}{(\zeta\omega_n)^2 - \omega_d^2} = \frac{\omega_n^2}{\zeta^2\omega_n^2 - \omega_n^2(\zeta^2-1)} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = \frac{\omega_n^2}{S\left[(S+\zeta\omega_n)^2 - \omega_d^2\right]} \times (S+\zeta\omega_n+\omega_d)\Big|_{S=-\zeta\omega_n-\omega_d}$$

$$= \frac{\omega_n^2}{S(S+\zeta\omega_n-\omega_d)}\Big|_{S=-\zeta\omega_n-\omega_d}$$

$$= \frac{\omega_n^2}{(-\zeta\omega_n-\omega_d)(-\zeta\omega_n-\omega_d+\zeta\omega_n-\omega_d)} = \frac{\omega_n^2}{(-\zeta\omega_n-\omega_d)(-2\omega_d)} = \frac{\omega_n^2}{2\omega_d(\zeta\omega_n+\omega_d)}$$

Scanned with CamScanner

$$= \frac{\omega_n^2}{2\omega_n \times \sqrt{\zeta^2-1} \times \left(\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)} \qquad \left[\because \omega_d = \omega_n\sqrt{\zeta^2-1}\right]$$

$$= \frac{\omega_n^2}{2\omega_n^2(\zeta^2-1) + 2\zeta\omega_n^2\sqrt{\zeta^2-1}} = \frac{1}{2\left[(\zeta^2-1) + \zeta\sqrt{\zeta^2-1}\right]}; \text{ suppose a}$$

Similarly;

$$C = \frac{\omega_n^2}{S\left[(S+\zeta\omega_n)^2 - \omega_d^2\right]} \times (S + \zeta\omega_n - \omega_d)\Big|_{S=-\zeta\omega_n+\omega_d}$$

$$= \frac{\omega_n^2}{S\left[S + \zeta\omega_n + \omega_d\right]}\Big|_{S=-\zeta\omega_n+\omega_d}$$

Solving; we get,

$$C = \frac{1}{2\left[(\zeta^2-1) - \zeta\sqrt{\zeta^2-1}\right]} = \frac{-1}{2\left[\zeta\sqrt{\zeta^2-1} - (\zeta^2-1)\right]}; \text{ suppose b}$$

Then,

$$C(S) = \frac{1}{S} + \frac{a}{(S+\zeta\omega_n+\omega_d)} + \frac{b}{(S+\zeta\omega_n-\omega_d)}$$

$$= \frac{1}{S} + \frac{a}{\left(S+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)} + \frac{b}{\left(S+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)}$$

$$= \frac{1}{S} + \frac{a}{S+\omega_n\left(\zeta+\sqrt{\zeta^2-1}\right)} + \frac{b}{S+\omega_n\left(\zeta-\sqrt{\zeta^2-1}\right)}$$

Taking inverse Laplace transform; we have,

$$C(t) = 1 + ae^{-\omega_n\left(\zeta+\sqrt{\zeta^2-1}\right)t} + be^{-\omega_n\left(\zeta-\sqrt{\zeta^2-1}\right)t}$$

Substituting values of a and b; we get,

$$C(t) = 1 + \frac{1}{2\left[(\zeta^2-1) + \zeta\sqrt{(\zeta^2-1)}\right]} e^{-\omega_n\left(\zeta+\sqrt{\zeta^2-1}\right)t} - \frac{1}{2\left[\zeta\sqrt{(\zeta^2-1)} - (\zeta^2-1)\right]} e^{-\omega_n\left(\zeta-\sqrt{\zeta^2-1}\right)t}$$

$$\therefore \quad C(t) = 1 + \frac{e^{-\omega_n\left(\zeta+\sqrt{\zeta^2-1}\right)t}}{2\left[(\zeta^2-1) + \zeta\sqrt{(\zeta^2-1)}\right]} - \frac{e^{-\omega_n\left(\zeta-\sqrt{\zeta^2-1}\right)t}}{2\left[\zeta\sqrt{(\zeta^2-1)} - (\zeta^2-1)\right]}$$

Also,

$$C(t) = 1 + \frac{e^{-\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}}{2\sqrt{(\zeta^2-1)}\left[\sqrt{\zeta^2-1}+\zeta\right]} - \frac{e^{-\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}}{2\sqrt{(\zeta^2-1)}\left[\zeta-\sqrt{\zeta^2-1}\right]}$$

Above response has two exponentially decaying terms,

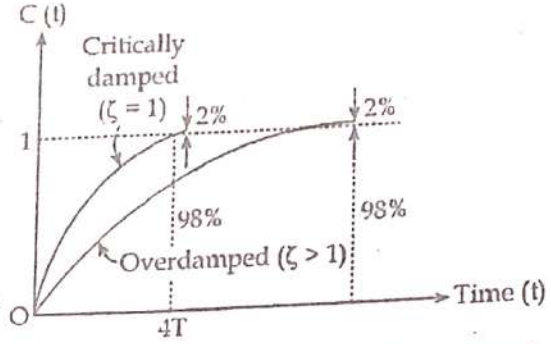
$$T_1 = \frac{1}{\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n}$$

and, $T_2 = \dfrac{1}{\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n}$

As first term (*i.e.*, T_1) is decaying faster [since, $T_1 < T_2$] than the second term, so first term can be neglected.

Scanned with CamScanner

$$\therefore \quad \boxed{C(t) = 1 - \frac{e^{-\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}}{2\sqrt{(\zeta^2 - 1)}\left[\zeta - \sqrt{\zeta^2 - 1}\right]}}$$



Graph shows that over damped is slower than critically damped.

iv) **Under-damped ($0 < \zeta < 1$) case**

Roots of characteristics equation are:

$$S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$$

or, $(S + \zeta\omega_n)^2 - \zeta^2\omega_n^2 + \omega_n^2 = 0$

or, $(S + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2) = 0$

Let, $\omega_d = \omega_n\sqrt{1 - \zeta^2}$ = Damped frequency of oscillation

Roots of equation are complex conjugate.

System oscillates (At reduced frequency compared to the undamped case) with amplitud gradually decreasing to zero.

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} = \frac{\omega_n^2}{\left[(S + \zeta\omega_n)^2 + \omega_d^2\right]}$$

For unit step i/p;

$$R(S) = \frac{1}{S}$$

Then,

$$C(S) = \frac{\omega_n^2}{S\left[(S + \zeta\omega_n)^2 + \omega_d^2\right]} = \frac{A}{S} + \frac{B}{(S + \zeta\omega_n)^2 + \omega_d^2}$$

Now,

$$A = \frac{\omega_n^2}{S\left[(S + \zeta\omega_n)^2 + \omega_d^2\right]} \times S\Big|_{S=0} = \frac{\omega_n^2}{(\zeta\omega_n)^2 + \omega_d^2} = \frac{\omega_n^2}{\zeta^2\omega_n^2 + \omega_n^2(1 - \zeta^2)} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = \frac{\omega_n^2}{S\left[(S + \zeta\omega_n)^2 + \omega_d^2\right]} \times \left[(S + \zeta\omega_n)^2 + \omega_d^2\right]\Big|_{S=-\zeta\omega_n - j\omega_d} = \frac{\omega_n^2}{-\zeta\omega_n - j\omega_d}$$

$$= \frac{-\omega_n^2}{\zeta\omega_n + j\omega_d} = \frac{-\omega_n^2}{(\zeta\omega_n + j\omega_d)} \times \frac{(\zeta\omega_n - j\omega_d)}{(\zeta\omega_n - j\omega_d)} = \frac{-\omega_n^2(\zeta\omega_n - j\omega_d)}{\zeta^2\omega_n^2 + \omega_d^2}$$

and, $\omega_d = \omega_n(1 - \zeta^2)$

so, $B = \dfrac{-\omega_n^2(\zeta\omega_n - j\omega_d)}{\zeta^2\omega_n^2 + \omega_n^2(1 - \zeta^2)} = \dfrac{-\omega_n^2(\zeta\omega_n - j\omega_d)}{\omega_n^2} = -(\zeta\omega_n - j\omega_d)$

We have,

Scanned with CamScanner

$$S = -\zeta\omega_n - j\omega_d$$

so, $$B = -\zeta\omega_n + j\omega_d = -\zeta\omega_n + (-S - \zeta\omega_n) = -S - 2\zeta\omega_n = -(S + 2\zeta\omega_n)$$

$$\therefore \quad C(S) = \frac{1}{S} - \frac{(S + 2\zeta\omega_n)}{(S + \zeta\omega_n)^2 + \omega_d^2} = \frac{1}{S} - \left[\frac{(S + \zeta\omega_n)}{(S + \zeta\omega_n)^2 + \omega_d^2} + \frac{\zeta\omega_n}{(S + \zeta\omega_n)^2 + \omega_d^2}\right]$$

$$= \frac{1}{S} - \left[\frac{(S + \zeta\omega_n)}{(S + \zeta\omega_n)^2 + \omega_d^2} + \frac{\zeta\omega_n}{\omega_d} \times \frac{\omega_d}{(S + \zeta\omega_n)^2 + \omega_d^2}\right]$$

Taking inverse Laplace transform; we get,

$$C(t) = 1 - \left[e^{-\zeta\omega_n t}\cos\omega_d t + e^{-\zeta\omega_n t}\sin\omega_d t\left(\frac{\zeta\omega_n}{\omega_d}\right)\right]$$

since, $$L(e^{-at}\cos bt) = \frac{(S + a)}{(S + a)^2 + b^2}$$

and, $$L(e^{-at}\sin bt) = \frac{b}{(S + a)^2 + b^2}$$

so, $$C(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \left(\frac{\zeta\omega_n}{\omega_d}\right)\sin\omega_d t\right] = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \left(\frac{\zeta\omega_n}{\omega_d\sqrt{1-\zeta^2}}\right)\sin\omega_d t\right]$$

$$= 1 - e^{-\zeta\omega_n t}\left(\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right)$$

Put, $\cos\phi = \zeta$

Then,

$$\sin\phi = \sqrt{1-\zeta^2}$$

so, $$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\sin\phi\cos\omega_d t + \cos\phi\sin\omega_d t)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}[\sin(\phi + \omega_d t)]$$

and, $$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\therefore \quad \phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\therefore \quad \boxed{C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left[\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) + \left(\omega_n\sqrt{1-\zeta^2}\right)t\right]}$$

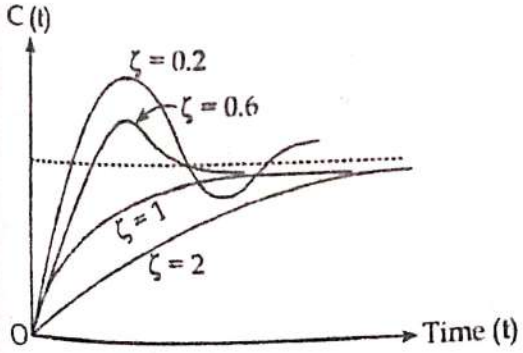


Figure: Response of underdaped, critically damped and overdamped

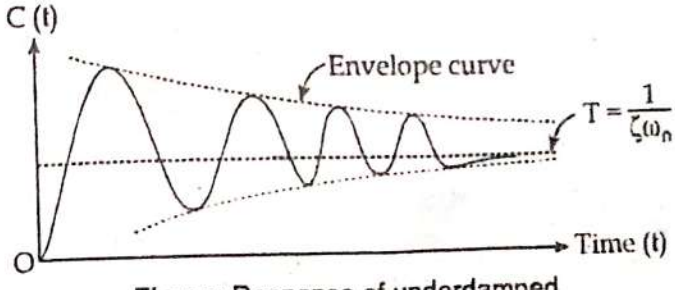


Figure: Response of underdamped

Scanned with CamScanner

ii) Time response of second order system with unit ramp input

Transfer function of second order system is;

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

For unit ramp input;

$$R(S) = \frac{1}{S^2}$$

$$\therefore \quad C(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} \times R(S) = \frac{\omega_n^2}{S^2\left(S^2 + 2\zeta\omega_n S + \omega_n^2\right)}$$

Partial fraction gives,

$$\frac{\omega_n^2}{S^2\left(S^2 + 2\zeta\omega_n S + \omega_n^2\right)} = \frac{A}{S} + \frac{B}{S^2} + \frac{CS + D}{\left(S^2 + 2\zeta\omega_n S + \omega_n^2\right)}$$

On solving; we have,

$$\frac{\omega_n^2}{S^2\left(S^2 + 2\zeta\omega_n S + \omega_n^2\right)} = \frac{AS\left(S^2 + 2\zeta\omega_n S + \omega_n^2\right) + B\left(S^2 + 2\zeta\omega_n S + \omega_n^2\right) + S^2(CS + D)}{S^2\left(S^2 + 2\zeta\omega_n S + \omega_n^2\right)}$$

or, $\omega_n^2 = AS\left(S^2 + 2\zeta\omega_n S + \omega_n^2\right) + B\left(S^2 + 2\zeta\omega_n S + \omega_n^2\right) + (CS + D)S^2$

Equating the coefficient of S^3; (1)

$$0 = A + C$$

Equating the coefficient of S^2; (2)

$$0 = 2\zeta\omega_n A + B + D$$

Equating the coefficient of S;

$$0 = A\omega_n^2 + 2\zeta\omega_n B \quad (3)$$

or, $0 = A\omega_n + 2\zeta B$

Equating the absolute term,

$$\omega_n^2 = B\omega_n^2$$

$$\therefore \quad \boxed{B = 1}$$

Substituting in equation (3); we have,

$$0 = A\omega_n + 2\zeta$$

$$\therefore \quad \boxed{A = \frac{-2\zeta}{\omega_n}}$$

Substituting in equation (1); we have,

$$0 = A + C$$

$$\therefore \quad \boxed{C = \frac{2\zeta}{\omega_n}}$$

Substituting in equation (2); we have,

$$0 = 2\zeta\omega_n \times A + B + D = 2\zeta\omega_n \times \left(\frac{-2\zeta}{\omega_n}\right) + 1 + D$$

or, $0 = -4\zeta^2 + 1 + D$

$$\therefore \quad \boxed{D = 4\zeta^2 - 1}$$

Substituting the value of A, B, C and D; we get,

Scanned with CamScanner

$$\frac{\omega_n^2}{S^2\left(S^2+2\zeta\omega_n S+\omega_n^2\right)} = \frac{-2\zeta}{\omega_n S^2}+\frac{1}{S^2}+\frac{\frac{2\zeta}{\omega_n}S+\left(4\zeta^2-1\right)}{\left(S^2+2\zeta\omega_n S+\omega_n^2\right)}$$

$$= \frac{1}{S^2}-\frac{2\zeta}{\omega_n S}+\frac{2\zeta S+\omega_n\left(4\zeta^2-1\right)}{\omega_n\left(S^2+2\zeta\omega_n S+\omega_n^2\right)}$$

$$= \frac{1}{S^2}-\frac{2\zeta}{\omega_n S}+\frac{2\zeta S}{\omega_n\left(S^2+2\zeta\omega_n S+\omega_n^2\right)}+\frac{\left(4\zeta^2-1\right)}{\left(S^2+2\zeta\omega_n S+\omega_n^2\right)}$$

$$= \frac{1}{S^2}-\frac{2\zeta}{\omega_n S}+\frac{2\zeta(S+\zeta\omega_n-\zeta\omega_n)}{\omega_n\left(S^2+2\zeta\omega_n S+\omega_n^2\right)}+\frac{\left(4\zeta^2-1\right)}{\left[(S+\zeta\omega_n)^2-\zeta^2\omega_n^2+\omega_n^2\right]}$$

$$= \frac{1}{S^2}-\frac{2\zeta}{\omega_n S}+\frac{2\zeta(S+\zeta\omega_n)}{\omega_n\left(S^2+2\zeta\omega_n S+\omega_n^2\right)}-\frac{2\zeta}{\omega_n}\frac{\zeta\omega_n}{\left(S^2+2\zeta\omega_n S+\omega_n^2\right)}$$

$$+\frac{\left(4\zeta^2-1\right)}{\left[(S+\zeta\omega_n)^2-\zeta^2\omega_n^2+\omega_n^2\right]}$$

$$= \frac{1}{S^2}-\frac{2\zeta}{\omega_n S}+\frac{2\zeta(S+\zeta\omega_n)}{\omega_n\left(S^2+2\zeta\omega_n S+\omega_n^2\right)}-\frac{2\zeta^2}{S^2+2\zeta\omega_n S+\omega_n^2}$$

$$+\frac{\left(4\zeta^2-1\right)}{\left[(S+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)\right]}$$

$$= \frac{1}{S^2}-\frac{2\zeta}{\omega_n S}+\frac{2\zeta(S+\zeta\omega_n)}{\omega_n\left(S^2+2\zeta\omega_n S+\omega_n^2\right)}+\frac{\left(4\zeta^2-2\zeta^2-1\right)}{S^2+2\zeta\omega_n S+\omega_n^2}$$

$$\therefore \quad C(S) = \frac{1}{S^2}-\frac{2\zeta}{\omega_n S}+\frac{2\zeta(S+\zeta\omega_n)}{\omega_n\left(S^2+2\zeta\omega_n S+\omega_n^2\right)}+\frac{\left(2\zeta^2-1\right)}{(S+\zeta\omega_n)^2+\omega_n^2\left(1-\zeta^2\right)}$$

We know that;

$$\omega_d^2 = \omega_n^2(1-\zeta^2)$$

$$\therefore \quad C(S) = \frac{1}{S^2}-\frac{2\zeta}{\omega_n S}+\frac{2\zeta}{\omega_n}\times\frac{(S+\zeta\omega_n)}{(S+\zeta\omega_n)^2+\omega_d^2}+\frac{\left(2\zeta^2-1\right)}{(S+\zeta\omega_n)^2+\omega_d^2}\times\frac{\omega_d}{\omega_d}$$

$$= \frac{1}{S^2}-\frac{2\zeta}{\omega_n S}+\frac{2\zeta}{\omega_n}\times\frac{(S+\zeta\omega_n)}{\left[(S+\zeta\omega_n)^2+\omega_d^2\right]}+\frac{\left(2\zeta^2-1\right)}{\omega_d}\frac{\omega_d}{\left[(S+\zeta\omega_n)^2+\omega_d^2\right]}$$

Taking inverse Laplace; we have,

$$C(t) = t-\frac{2\zeta}{\omega_n}+\frac{2\zeta}{\omega_n}e^{-\zeta\omega_n t}\cos\omega_d t+\left(\frac{2\zeta^2-1}{\omega_d}\right)e^{-\zeta\omega_n t}\sin\omega_d t$$

$$= t-\frac{2\zeta}{\omega_n}+e^{-\zeta\omega_n t}\times\frac{2\zeta}{\omega_n}\times\frac{\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}}\times\cos\omega_d t+\frac{2\zeta^2-1}{\omega_n\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t$$

$$= t-\frac{2\zeta}{\omega_n}+\frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\left[2\zeta\sqrt{1-\zeta^2}\cos\omega_d t+(2\zeta^2-1)\sin\omega_d t\right]$$

Put, $\sin\phi = 2\zeta\sqrt{1-\zeta^2}$; then,

Scanned with CamScanner

$$\cos\phi = \sqrt{1-\sin^2\phi} = \sqrt{1-4\zeta^2(1-\zeta^2)} = \sqrt{1-4\zeta^2+4\zeta^4}$$

$$= \sqrt{(2\zeta^2-1)^2} = (2\zeta^2-1)$$

So, the equation becomes;

$$C(t) = t - \frac{2\zeta}{\omega_n} + \frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}[\sin\phi\cos\omega_d t + \cos\phi\sin\omega_d t]$$

and, $\tan\phi = \dfrac{\sin\phi}{\cos\phi} = \dfrac{2\zeta\sqrt{1-\zeta^2}}{2\zeta^2-1}$

or, $\phi = \tan^{-1}\left(\dfrac{2\zeta\sqrt{1-\zeta^2}}{2\zeta^2-1}\right)$

So, the response becomes;

$$C(t) = t - \frac{2\zeta}{\omega_n} + \frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\sin(\phi+\omega_d t)$$

$$\therefore \quad \boxed{C(t) = t - \frac{2\zeta}{\omega_n} + \frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t + \tan^{-1}\left(\frac{2\zeta\sqrt{1-\zeta^2}}{2\zeta^2-1}\right)\right]}$$

Error signal;

$$e(t) = r(t) - C(t)$$

and, r(t) = Ramp input = t

$$\therefore \quad \text{Error} = e(t) = t - t + \frac{2\zeta}{\omega_n} - \frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right]$$

$$\therefore \quad \text{Error } e(t) = \frac{2\zeta}{\omega_n} - \frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t + \tan^{-1}\left(\frac{2\zeta\sqrt{1-\zeta^2}}{2\zeta^2-1}\right)\right]$$

When $t \to \infty$; gives steady state error (e_{ss})

$$\boxed{e_{ss} = \lim_{t\longrightarrow\infty} e(t) = \frac{2\zeta}{\omega_n}}$$

iii) Response of second order system with unit impulse

Transfer function of second order system is;

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

For unit impulse;

$$R(S) = 1$$

$$\therefore \quad C(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} \times R(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

By partial fraction;

$$\frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} = \frac{\omega_n^2}{(S+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} = \frac{\omega_n^2}{(S+\zeta\omega_n)^2 - j^2\omega_d^2}$$

Characteristics equation of second degree equation is;

$$S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$$

or, $(S+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2) = 0$

Scanned with CamScanner

or, $(S + \zeta\omega_n)^2 = -\omega_n^2(1 - \zeta^2)$

or, $S = \zeta\omega_n \pm j\omega_n\left(\sqrt{1 - \zeta^2}\right)$

or, $S_1 = -\zeta\omega_n + j\omega_n\left(\sqrt{1 - \zeta^2}\right) = -\zeta\omega_n + j\omega_d$

and, $S_2 = -\zeta\omega_n - j\omega_n\left(\sqrt{1 - \zeta^2}\right) = -\zeta\omega_n - j\omega_d$

Partial fraction becomes;

$$\frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} = \frac{K_1}{(S + \zeta\omega_n + j\omega_d)} + \frac{K_2}{(S + \zeta\omega_n - j\omega_d)}$$

and, $K_1 = \dfrac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} \times (S + \zeta\omega_n + j\omega_d)\Big|_{S=-\zeta\omega_n - j\omega_d}$

$$= \frac{\omega_n^2}{(S + \zeta\omega_n + j\omega_d)(S + \zeta\omega_n - j\omega_d)} \times (S + \zeta\omega_n + j\omega_d)\Big|_{S=-\zeta\omega_n - j\omega_d}$$

$$= \frac{\omega_n^2}{(-\zeta\omega_n - j\omega_d + \zeta\omega_n - j\omega_d)} = -\frac{\omega_n^2}{2j\omega_d}$$

and, $K_2 = \dfrac{\omega_n^2}{(S + \zeta\omega_n + j\omega_d)(S + \zeta\omega_n - j\omega_d)} \times (S + \zeta\omega_n - j\omega_d)\Big|_{S=-\zeta\omega_n + j\omega_d}$

$$= \frac{\omega_n^2}{(S + \zeta\omega_n + j\omega_d)}\Big|_{S=-\zeta\omega_n + j\omega_d} = \frac{\omega_n^2}{(-\zeta\omega_n + j\omega_d + \zeta\omega_n + j\omega_d)} = \frac{\omega_n^2}{2j\omega_d}$$

So, the equation becomes;

$$C(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} = \frac{\omega_n^2}{2j\omega_d(S + \zeta\omega_n - j\omega_d)} - \frac{\omega_n^2}{2j\omega_d(S + \zeta\omega_n + j\omega_d)}$$

$$\therefore \quad C(S) = \frac{\omega_n^2}{2j\omega_d}\left[\frac{1}{S + (\zeta\omega_n - j\omega_d)} - \frac{1}{S + (\zeta\omega_n + j\omega_d)}\right]$$

Taking inverse Laplace transform; we have,

$$C(t) = \frac{\omega_n^2}{2j\omega_d}\left[e^{-(\zeta\omega_n - j\omega_d)t} - e^{-(\zeta\omega_n + j\omega_d)t}\right] = \frac{\omega_n^2}{\omega_d}\left[\frac{e^{-\zeta\omega_n t} \cdot e^{j\omega_d t} - e^{-\zeta\omega_n t} \cdot e^{-j\omega_d t}}{2j}\right]$$

$$= \frac{\omega_n^2 e^{-\zeta\omega_n t}}{\omega_d}\left[\frac{e^{j\omega_d t} - e^{-j\omega_d t}}{2j}\right]$$

We know that;

$$\sin\omega_d t = \frac{e^{j\omega_d t} - e^{-j\omega_d t}}{2j}$$

so, $C(t) = \dfrac{\omega_n^2}{\omega_n\sqrt{1 - \zeta^2}} \times e^{-\zeta\omega_n t} \times \sin\omega_d t$

$$\therefore \quad \boxed{C(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} \times e^{-\zeta\omega_n t} \times \sin\left(\omega_n\sqrt{1 - \zeta^2}\right)t}$$

i) For critically damped condition

$\zeta = 1$

$$\therefore \quad C(t) = \lim_{\zeta \longrightarrow 1} \frac{\omega_n}{\sqrt{1 - \zeta^2}} \times e^{-\zeta\omega_n t} \times \sin\left(\omega_n\sqrt{1 - \zeta^2}\right)t$$

Solving; we get,

Scanned with CamScanner

$$C(t) = te^{-\omega_n t}\omega_n^2$$

ii) For under damped $(0 < \zeta < 1)$

$$C(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} \times e^{-\zeta\omega_n t} \times \sin\left(\omega_n\sqrt{1-\zeta^2}\right)t$$

iii) For over-damped $(\zeta > 1)$

$$C(t) = \omega_n e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}} \times \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

3.4.4 Transient Response Characteristics/Specifications

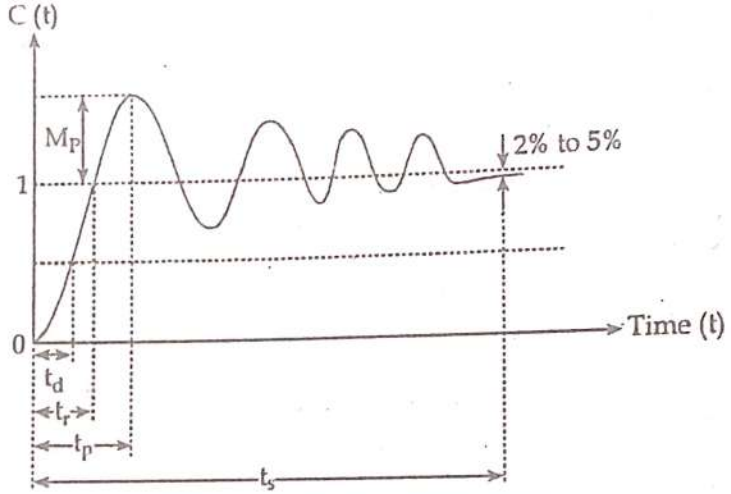


Figure: Underdamped response

i) Delay time (t_d)

Time required for response to reach 50% and of final value in first time; we have,

$$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left[\omega_n\sqrt{1-\zeta^2}t + \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right]$$

If $C(t)$, ζ and ω_n is known; then, putting $C(t) = 50\%$ and of final value will give t_d.

ii) Rise time (t_r)

Time required for response to reach from 0 to 100% for under damped and 10% to 90% of its final value for over-damped.

$$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left[\omega_n\sqrt{1-\zeta^2}t + \phi\right]$$

where, $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$.

Now,

$$C(t) = 100\% = 1$$

Then,

$$1 = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left[\omega_n\sqrt{1-\zeta^2}t + \phi\right]$$

or, $$\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left[\omega_n\sqrt{1-\zeta^2}t + \phi\right] = 0$$

or, $$e^{-\zeta\omega_n t} \neq 0$$

Scanned with CamScanner

so, $\sin\left[\omega_n\sqrt{1-\zeta^2}t+\phi\right]=0$

or, $\sin\left[\omega_n\sqrt{1-\zeta^2}t+\phi\right]=0=\sin n\pi$

where, $n=1, 2, 3, 4, \ldots\ldots$

$$\omega_n\sqrt{1-\zeta^2}t+\phi=n\pi$$

$$\therefore \quad t=\frac{n\pi-\phi}{\omega_n\sqrt{1-\zeta^2}}$$

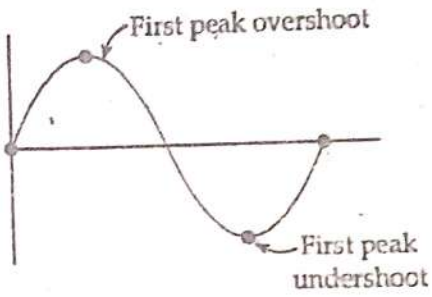
For rise time; put $n=1$ (First peak);

$$t_r=\frac{n\pi-\phi}{\omega_n\sqrt{1-\zeta^2}}=\frac{\pi-\phi}{\omega_n\sqrt{1-\zeta^2}} \text{ [use } \phi \text{ in radian]}$$

$$\therefore \quad t_r=\frac{\pi-\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}{\omega_n\sqrt{1-\zeta^2}}$$; which is the expression for rise time t_r.

iii) Peak time (t_p)

Time required for response to reach first peak of time response or first peak overshoot.



$$C(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left[\omega_n\sqrt{1-\zeta^2}t+\phi\right]$$

For maxima;

$$\frac{d[C(t)]}{dt}=0$$

Using product rule of derivative,

$$-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos\left[\omega_n\sqrt{1-\zeta^2}t_p+\phi\right]\omega_n\sqrt{1-\zeta^2}+\sin\left[\omega_n\sqrt{1-\zeta^2}t_p+\phi\right]\frac{\zeta\omega_n}{\sqrt{1-\zeta^2}}\times e^{-\zeta\omega_n t}=0$$

or, $$-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\omega_n\sqrt{1-\zeta^2}\cos\left(\omega_n\sqrt{1-\zeta^2}t_p+\phi\right)-\zeta\omega_n\sin\left(\omega_n\sqrt{1-\zeta^2}t_p+\phi\right)\right]=0$$

since, $e^{-\zeta\omega_n t}\neq 0$

so, $$\sqrt{1-\zeta^2}\cos\left(\omega_n\sqrt{1-\zeta^2}t_p+\phi\right)-\zeta\sin\left(\omega_n\sqrt{1-\zeta^2}t_p+\phi\right)=0$$

or, $$\sqrt{1-\zeta^2}\cos\left(\omega_n\sqrt{1-\zeta^2}t_p+\phi\right)=\zeta\sin\left(\omega_n\sqrt{1-\zeta^2}t_p+\phi\right)$$

Put, $\zeta=\cos\phi$; then,

$$\sin\phi=\sqrt{1-\zeta^2}$$

and, $$\sin\phi\cos\left(\omega_n\sqrt{1-\zeta^2}t_p+\phi\right)-\cos\phi\sin\left(\omega_n\sqrt{1-\zeta^2}t_p+\phi\right)=0$$

or, $$\sin\left[\phi-\left(\omega_n\sqrt{1-\zeta^2}t_p+\phi\right)\right]=0$$

or, $$\sin\left(-\omega_n\sqrt{1-\zeta^2}t_p\right)=0$$

or, $$-\sin\left(\omega_n\sqrt{1-\zeta^2}t_p\right)=0$$

or, $$\sin\left(\omega_n\sqrt{1-\zeta^2}t_p\right)=0=\sin n\pi$$

where, $n=1, 2, 3, 4, \ldots\ldots$

Scanned with CamScanner

and, $\sin\left(\omega_n\sqrt{1-\zeta^2}t_p\right) = \sin n\pi$

or, $\omega_n\sqrt{1-\zeta^2}t_p = \sin n\pi$

$\therefore$ $\boxed{t_p = \dfrac{n\pi}{\omega_n\sqrt{1-\zeta^2}}}$

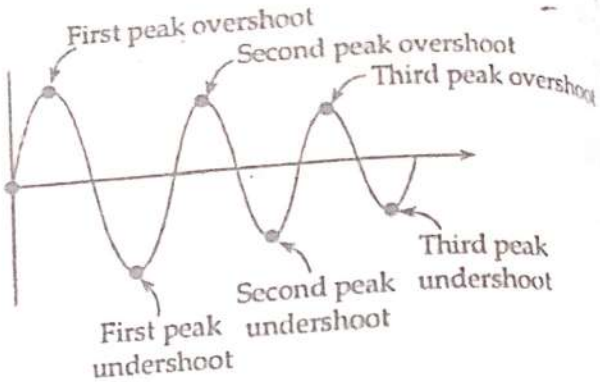
For first peak overshoot; n = 1

$\boxed{t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$;

which is the expression for peak time.

i.e., n = 1 = First peak overshoot
n = 2 = First peak undershoot
n = 3 = Second peak overshoot
n = 4 = Second peak undershoot

and so on.



iv) Maximum overshoot (M_p)

Maximum overshoot occurs at $t = t_p$.

Setting time occurs at $t = \infty$.

$\therefore$ Maximum percent overshoot $= \dfrac{C_{(t=t_p)} - C_{(t=\infty)}}{C_{t=\infty}} \times 100\%$

Response at $t = \infty$ (*i.e.*, settling time response)

$C_{(t=\infty)} = 1$

Now, we have to calculate response at peak time *i.e.*, $\left(C_{t_p}\right)$.

We have,

$$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left[\omega_n\sqrt{1-\zeta^2}t_p + \phi\right]$$

Put, $t = t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

so, $$C\left(t = t_p\right) = 1 - \frac{e^{-\zeta\omega_n \times \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\sin\left[\omega_n\sqrt{1-\zeta^2} \times \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} + \phi\right]$$

$$= 1 - \frac{e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}}{\sqrt{1-\zeta^2}}\sin(\pi + \phi) = 1 + \frac{e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}}{\sqrt{1-\zeta^2}}\sin\phi$$

$$= 1 + \frac{e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}}{\sqrt{1-\zeta^2}}\sqrt{1-\zeta^2} = 1 + e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$$

$\therefore$ Maximum percentage overshoot $\left(M_p\right) = \dfrac{C_{(t=t_p)} - C_{(t=\infty)}}{C_{t=\infty}} \times 100\% = \dfrac{1 + e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} - 1}{1} \times 100\%$

$\therefore$ $\boxed{M_p = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} \times 100\%}$

Scanned with CamScanner

v) **Settling time (t_s)**

Time required for response to reach and stay within the specified range of 2% to 5% of final value,

$$\boxed{t_s = 4T = \frac{4}{\zeta\omega_n}}$$

3.5 EFFECT OF FEEDBACK ON STEADY STATE GAIN, BANDWIDTH, ERROR MAGNITUDE, SYSTEM DYNAMICS OR RESPONSE, STABILITY, AND PARAMETER VARIATION

3.5.1 Effect of Feedback on System Dynamics or Response

Why closed loop system is faster than open loop system?

Open loop system does not have feedback but closed loop system have. This can be illustrated by following expressions:

i) For open loop system

$$R(S) \longrightarrow \boxed{\frac{1}{ST+1}} \longrightarrow C(S);\ \text{for first order system}$$

For unit step input;

$$R(S) = \frac{1}{S}$$

$$\therefore \quad C(S) = \frac{1}{ST+1} \times R(S) = \frac{1}{ST+1} \times \frac{1}{S} = \frac{1}{S(ST+1)} = \frac{1}{S} - \frac{T}{ST+1}$$

Taking inverse Laplace; we have,

$$C(t) = 1 - e^{-\frac{t}{T}} \qquad (1)$$

where, T is the time constant of first order system.

See laplace transform and inverse.

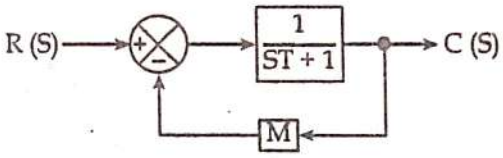
ii) **For closed loop system**

Now, transfer function becomes;

$$T.F. = \frac{C(S)}{R(S)} = \frac{\frac{1}{ST+1}}{1 + M\left(\frac{1}{ST+1}\right)} = \frac{1}{ST+1+M}$$

$$= \frac{1}{T\left[S + \left(\frac{1+M}{T}\right)\right]} = \frac{\frac{1}{T}}{S + \left(\frac{1+M}{T}\right)}$$

For first order system



and, for unit step input; $R(S) = \frac{1}{S}$

$$\therefore \quad C(S) = T.F. \times R(S) = \frac{1}{S} \times \frac{\frac{1}{T}}{S + \left(\frac{1+M}{T}\right)} = \frac{1}{T}\left[\frac{1}{S} - \frac{\frac{1+M}{T}}{S + \left(\frac{1+M}{T}\right)}\right]$$

Taking inverse Laplace; we have,

$$C(t) = \frac{1}{T}\left[1 - e^{-\left(\frac{1+M}{T}\right)t}\right] \qquad (2)$$

Now, comparing equation **(1)** and **(2)**; we have,

Time constant of open loop system = T

Time constant of closed loop system = $T' = \left(\frac{T}{1+M}\right)$

so, $T' < T$

Therefore, system response becomes fast with feedback. *i.e.*, **CLCS are faster than OLCS.**

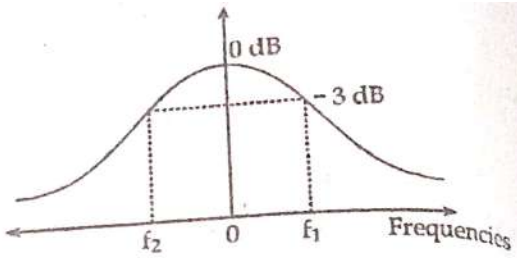
3.5.2 Effect of Feedback on Bandwidth

Bandwidth is defined as range of frequencies that belongs to **– 3 dB gain.**

Scanned with CamScanner

In the figure; we have,

Bandwidth (Bw) $= |f_2 - f_1| = |f_1 - f_2|$



For particular first order system;

$$f_{3\,dB} \times t_r = 0.35$$

where, t_r is the rise time.

$$f_{3\,dB} = \frac{0.35}{t_r}$$

or, $$f_{3\,dB} \propto \frac{1}{t_r}$$

Bandwidth frequency is inversely proportional to rise time.

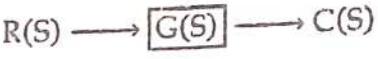
When feedback is implemented, time constant of the system decreases, *i.e.*, system becomes fast which means rise time (t_r) decreases, leads to increase in bandwidth frequency.

Therefore, closed loop system increases bandwidth.

3.5.3 Effect of Feedback on Steady Gain

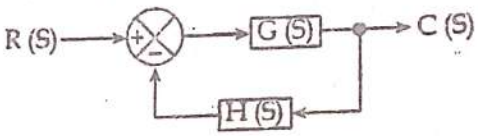
The overall transfer function of open loop system is;

$$\frac{C(S)}{R(S)} = G(S)$$



The overall transfer function of closed loop system is;

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)\,H(S)}$$



For negative feedback, the gain G(S) is reduced by a factor $\frac{1}{1 + G(S)\,H(S)}$. **So, due to negative feedback overall gain of the system reduces.**

3.5.4 Effect of Feedback on Stability

Consider the open loop system;

$$G(S) = \frac{K}{S + T}$$

The pole is located at $S = -T$.

Consider a closed loop system with unity negative feedback;

$$\frac{C(S)}{R(S)} = \frac{\frac{1}{S+T}}{1 + \frac{K}{S+T}} = \frac{K}{S + (T + K)}$$

The pole is located at $S = -(T + K)$.

Location of poles determines the stability. Therefore, feedback may be harmful or may improve the stability.

3.5.5 Effect of Parameter Variations (Robustness)

In closed loop system;

$$\frac{C(S)}{R(S)} = G(S)$$

Let, $\Delta G(S)$ is the change in $G(S)$ due to parameter variations.

and, $\Delta C(S)$ is the corresponding change in output.

$\therefore$ $C(S) = R(S) \cdot G(S)$

$$C(S) + \Delta C(S) = R(S)[G(S) + \Delta G(S)]$$

$$C(S) + \Delta C(S) = R(S) \cdot G(S) + R(S) \cdot \Delta G(S)$$

Scanned with CamScanner

$$C(S) + \Delta C(S) = C(S) + R(S) \cdot \Delta G(S)$$
$$\Delta C(S) = \Delta G(S) \cdot R(S) \quad (1)$$

Equation (1) gives the change in output due to parameter variations in open control system.

In closed loop system;

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)\,H(S)}$$

or, $$C(S) = \frac{G(S)}{1 + G(S)\,H(S)} \cdot R(S)$$

If $\Delta G(S)$ is the change due to parameter variation; then,

$$C(S) + \Delta C(S) = \frac{G(S) + \Delta G(S)}{1 + [G(S) + \Delta G(S)]H(S)} \cdot R(S) = \frac{[G(S) + \Delta G(S)]R(S)}{1 + G(S)H(S) + \Delta G(S)H(S)}$$

since, $\Delta G(S)H(S) \ll [1 + G(S)H(S)]$; neglecting

$$C(S) + \Delta C(S) = \frac{[G(S) + \Delta G(S)]R(S)}{1 + G(S)H(S)} = \frac{G(S)}{1 + G(S)H(S)} \cdot R(S) + \frac{\Delta G(S)}{1 + G(S)H(S)} \cdot R(S)$$

or, $$C(S) + \Delta C(S) = C(S) + \frac{\Delta G(S)}{1 + G(S)H(S)} \cdot R(S)$$

$$\therefore \quad \Delta C(S) = \frac{\Delta G(S)}{1 + G(S)H(S)} \cdot R(S) \quad (2)$$

Generally;

$$[G(S)H(S)] \gg 1$$

So, change in output is reduced due to parameter variations in G(S) by $[1 + G(S) \cdot H(S)]$. But, in open loop system there is no reduction because of no feedback.

3.5.6 Effect of Feedback on Error Magnitude (Accuracy)

Feedback reduces error magnitude in general. Therefore, closed loop control systems are more accurate than open loop system.

The error signal is defined as the difference of desired output and actual output. The controller regulates the error signal form error detector. The output transducer senses the actual output of the system and passes it to the error detector in regulated format. The output transducer here acts as a feedback. As feedback continuously maintaining the actual output at desirable output, such systems are more reliable, accurate and faster as well.

Thus, feedback mechanism plays pivotal role in control system which ultimately provides optimization.

3.5.7 Effect of Disturbances on Control System

In open loop control system;

$$C(S) = G(S) \cdot R(S)$$

Due to disturbances, there will be change in G(S); so, corresponding change in output is;

$$\Delta C(S) = \Delta G(S) \cdot R(S) \quad (1)$$

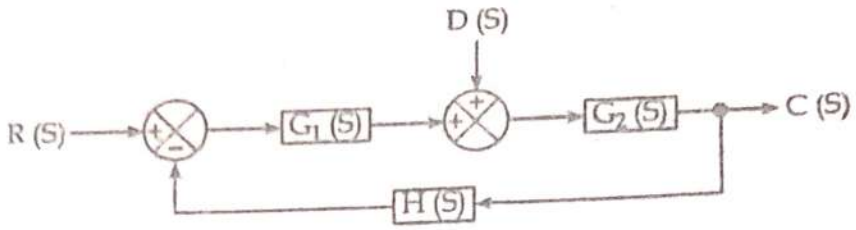
In closed loop control system, corresponding change in output is;

$$\Delta C(S) = \frac{\Delta G(S)}{1 + G(S)H(S)} \cdot R(S) \quad (2)$$

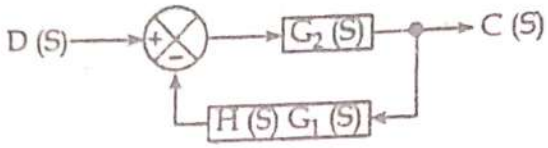
Hence, the fluctuation in output can be reduced by increasing the value of both G(S) and H(S). Hence, the closed loop control systems have better disturbance rejection capability.

Clearly,

Scanned with CamScanner



For R(S) = 0; the above system reduces to;



$$\text{or,} \quad \frac{C(S)}{D(S)} = \frac{G_2(S)}{1 + G_1(S)\,G_2(S)H(S)}$$

If $G_1(S)G_2(S)H(S) \gg 1$; then equation (1) becomes;

$$\frac{C(S)}{D(S)} = \frac{1}{G(S)H(S)}$$

$$\text{or,} \quad C(S) = \frac{1}{G(S)H(S)} \cdot D(S)$$

where, $G(S) = G_1(S)G_2(S)$ = Forward T.F.

$H(S)$ = Backward T.F.

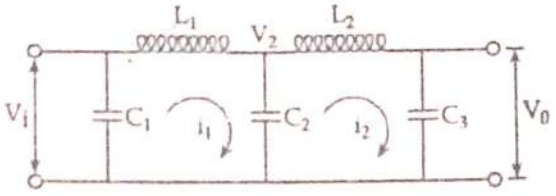
$D(S)$ = Distribution signal

Hence, effect of disturbance can be suppressed by increasing the gain of G(S) and [or H(S)].

Scanned with CamScanner

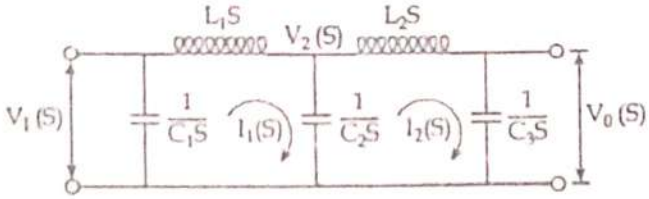
3.6 SOLVED PROBLEMS

1. Draw the block diagram representation of the circuit below. [2065 Chaitra]



Solution:

The circuit in S-domain becomes;



Here,

$$I_1(S) = \frac{V_1(S) - V_2(S)}{L_1 S} \quad (1)$$

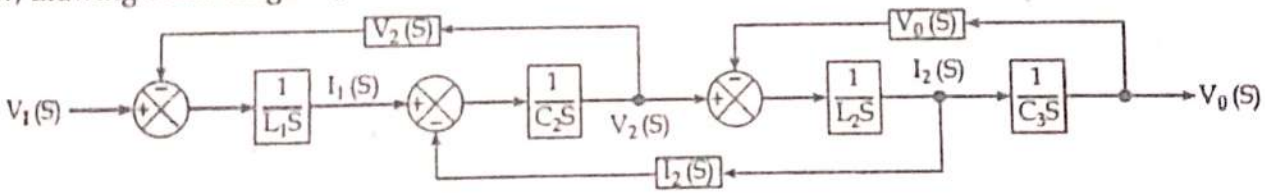
$$V_2(S) = \frac{1}{C_2 S}[I_1(S) - I_2(S)] \quad (2)$$

$$I_2(S) = \frac{V_2(S) - V_0(S)}{L_2 S} \quad (3)$$

$$V_0(S) = \frac{1}{C_3 S} I_2(S) \quad (4)$$

and, $$V_1(S) = \frac{1}{C_1 S} I_1(S) \quad (5)$$

Now, drawing block diagram;



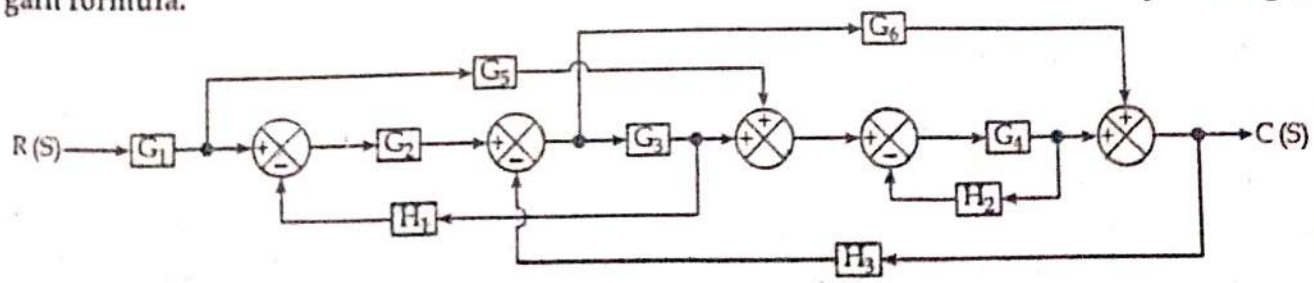
2. How can a response to any external disturbances injected to a plant be suppressed in closed loop control system? [2066 Magh]

Ans: By increasing the value of G(S)H(S) in closed loop control system.

For detail

See the definition part 3.5.7

3. Develop signal flow graph of the block diagram given and find transfer function using Mason gain formula. [2066 Magh]

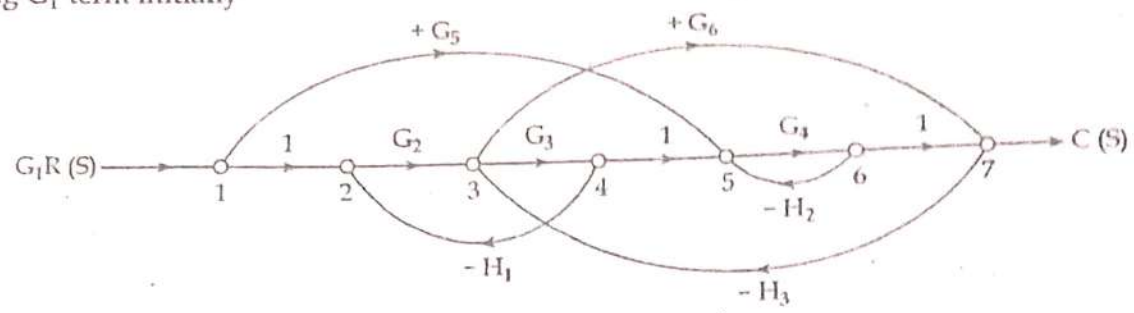


Scanned with CamScanner

Solution:

Drawing signal flow graph;

Neglecting G_1 term initially



Forward paths;

$g_1 = 1 \times G_2 \times G_3 \times 1 \times G_4 \times 1$ $[1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7]$

$= G_2G_3G_4$

$g_2 = G_5 \times G_4 \times 1 = G_4G_5$ $[1 \to 5 \to 6 \to 7]$

$g_3 = 1 \times G_2 \times G_6 = G_2G_6$ $[1 \to 2 \to 3 \to 7]$

Individual loops;

$L_1 = G_4 \times (-H_2) = -G_4H_2$ $[5 \to 6 \to 5]$

$L_2 = G_3 \times 1 \times G_4 \times 1 \times (-H_3) = -G_3G_4H_3$ $[3 \to 4 \to 5 \to 6 \to 7 \to 3]$

$L_3 = G_2 \times G_3 \times (-H_1) = -G_2G_3H_1$ $[2 \to 3 \to 4 \to 2]$

$L_4 = G_6 \times (-H_3) = -G_6H_3$ $[3 \to 7 \to 3]$

Now,

$\Delta = 1 - [L_1 + L_2 + L_3 + L_4] + [L_1L_3 + L_1L_4]$

$= 1 - [-G_4H_2 - G_3G_4H_3 - G_2G_3H_1 - G_6H_3] + [G_2G_3G_4H_1H_2 + G_4G_6H_2H_3]$

$= 1 + G_4H_2 + G_3G_4H_3 + G_2G_3H_1 + G_6H_3 + G_2G_3G_4H_1H_2 + G_4G_6H_2H_3$

Δ_1 = Part of Δ not touching g_1 path $= 1 - 0 = 1$

Δ_2 = Part of Δ not touching g_2 path $= 1 - L_3 = 1 + G_2G_3H_1$

Δ_3 = Part of Δ not touching g_3 path $= 1 - L_1 = 1 + G_4H_2$

From Masons gain formula; we have,

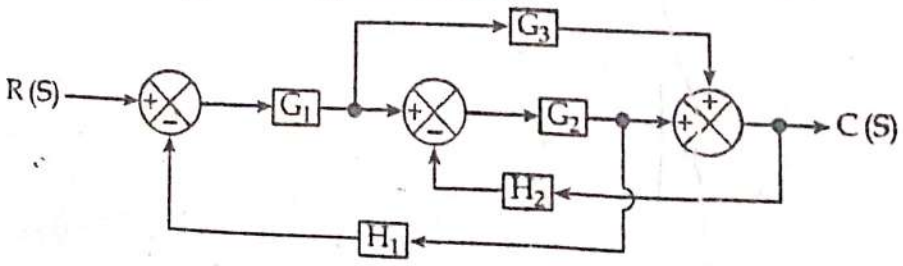
$$\text{T.F.} = \frac{C(S)}{R(S)} = \frac{\sum g_k\Delta_k}{\Delta} = \frac{g_1\Delta_1 + g_2\Delta_2 + g_3\Delta_3}{\Delta}$$

$$= \frac{G_2G_3G_4 + G_4G_5(1 + G_2G_3H_1) + G_2G_6(1 + G_4H_2)}{1 + G_4H_2 + G_3G_4H_3 + G_2G_3H_1 + G_6H_3 + G_2G_3G_4H_1H_2 + G_4G_6H_2H_3}$$

Considering G_1; we have,

$$\therefore \quad \boxed{\text{T.F.} = \frac{C(S)}{R(S)} = \frac{G_1[G_2G_3G_4 + G_4G_5(1 + G_2G_3H_1) + G_2G_6(1 + G_4H_2)]}{1 + G_4H_2 + G_3G_4H_3 + G_2G_3H_1 + G_6H_3 + G_2G_3G_4H_1H_2 + G_4G_6H_2H_3}}$$

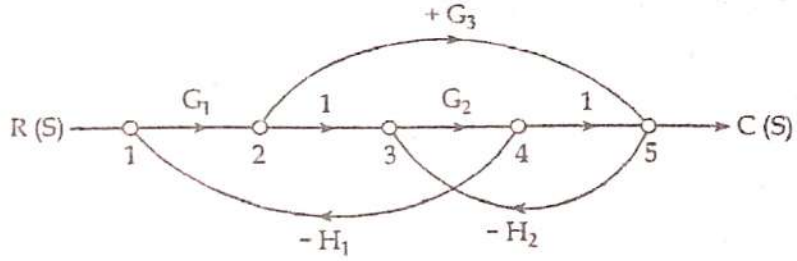
4. Find transfer function using Mason gain formula: **[2067 Ashadh]**



Solution:

Scanned with CamScanner

Drawing signal flow graph;



Now, forward paths are;

$g_1 = G_1 \times 1 \times G_2 \times 1 = G_1G_2$ $[1 \to 2 \to 3 \to 4 \to 5]$

$g_2 = G_1 \times G_3 = G_1G_3$ $[1 \to 2 \to 5]$

and, individual loops are;

$L_1 = G_2 \times 1 \times (-H_2) = -G_2H_2$ $[3 \to 4 \to 5 \to 3]$

$L_2 = G_1 \times 1 \times G_2 \times (-H_1) = -G_1G_2H_1$ $[1 \to 2 \to 3 \to 4 \to 1]$

$L_3 = G_1 \times G_3 \times (-H_2) \times G_2 \times (-H_1)$ $[1 \to 2 \to 5 \to 3 \to 4 \to 1]$

$= G_1G_2G_3H_1H_2$

$\Delta = 1 - [L_1 + L_2 + L_3] + [0]$

$= 1 - [-G_2H_2 - G_1G_2H_1 + G_1G_2G_3H_1H_2]$

$= 1 + G_2H_2 + G_1G_2H_1 - G_1G_2G_3H_1H_2$

Δ_1 = Part of Δ not touching g_1 path $= 1 - 0 = 1$

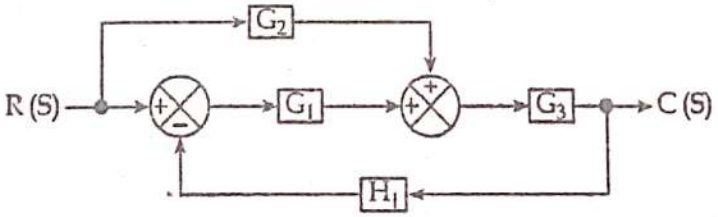
Δ_2 = Part of Δ not touching g_2 path $= 1 - 0 = 1$

From Mason gain formula; we have,

$$\text{T.F.} = \frac{C(S)}{R(S)} = \frac{\sum g_k \Delta_k}{\Delta} = \frac{g_1\Delta_1 + g_2\Delta_2}{\Delta} = \frac{G_1G_2 + G_1G_3}{1 + G_2H_2 + G_1G_2H_1 - G_1G_2G_3H_1H_2}$$

$$\therefore \quad \boxed{\text{T.F.} = \frac{C(S)}{R(S)} = \frac{G_1[G_2 + G_3]}{1 + G_2H_2 + G_1G_2H_1 - G_1G_2G_3H_1H_2}}$$

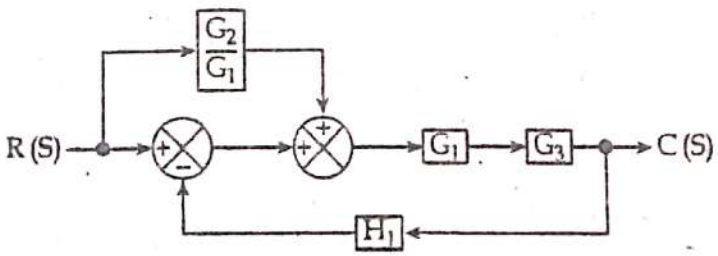
5. Reduce using reduction technique and derive transfer function. **[2067 Magh]**



Solution:

Step 1

Moving summing point ahead of G_1

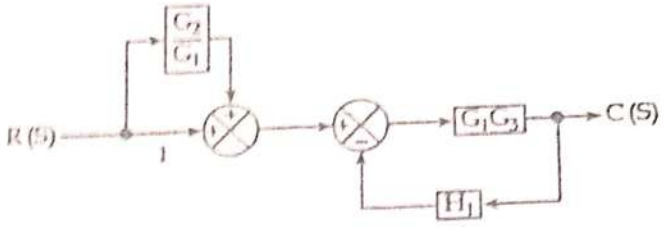


Step 2

i) Block in cascade G_1 and G_3

ii) And exchanging summing points

Scanned with CamScanner



Step 3

i) Blocks G_2 and 1 are in parallel

ii) Reducing negative feedback circuit

$$R(S) \longrightarrow \boxed{1+\frac{G_2}{G_1}} \longrightarrow \boxed{\frac{G_1G_3}{1+G_1G_3H_1}} \longrightarrow C(S)$$

Step 4

Block in cascade

$$\therefore \quad \text{Transfer function} = \frac{C(S)}{R(S)} = \left(1+\frac{G_2}{G_1}\right)\left(\frac{G_1G_3}{1+G_1G_3H_1}\right) = \frac{G_1G_3(G_1+G_2)}{G_1(1+G_1G_3H_1)}$$

$$\therefore \quad \boxed{\frac{C(S)}{R(S)} = \frac{G_3(G_1+G_2)}{(1+G_1G_3H_1)}}$$

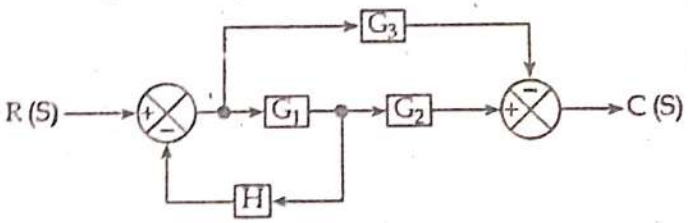
6. How introduction of feedback on system makes the system response faster? [2068 Baishakh]

Ans: See the definition part 3.5.1

7. Effect of disturbances can be reduced by increasing the gain of forward path. [2068 Baishakh]

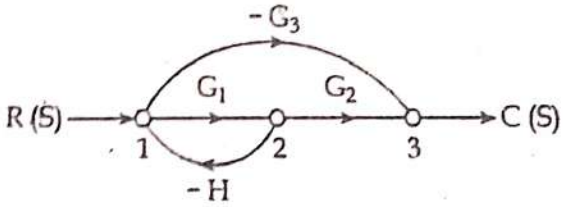
Ans: See the definition part 3.5.5 and 3.5.7

8. Find transfer function using Mason gain formula. [2069 Chaitra]



Solution:

Drawing S.F.G. of above system



Now, forward paths are;

$g_1 = G_1 \times G_2 = G_1G_2$ [1 → 2 → 3]

$g_2 = -(G_3)$ [1 → 3]

and, individual loops are;

$L_1 = G_1 \times (-H) = -G_1H$ [1 → 2 → 1]

$\Delta = 1 - L_1 = 1 + G_1H$

Δ_1 = Part of Δ not touching $g_1 = 1 - 0 = 1$

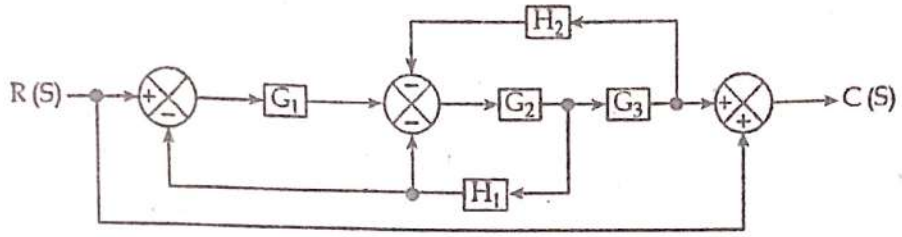
Δ_2 = Part of Δ not touching $g_2 = 1 - 0 = 1$

Scanned with CamScanner

Transfer function (T.F.) $= \frac{C(S)}{R(S)} = \frac{\sum g_k \Delta_k}{\Delta} = \frac{g_1\Delta_1 + g_2\Delta_2}{\Delta}$

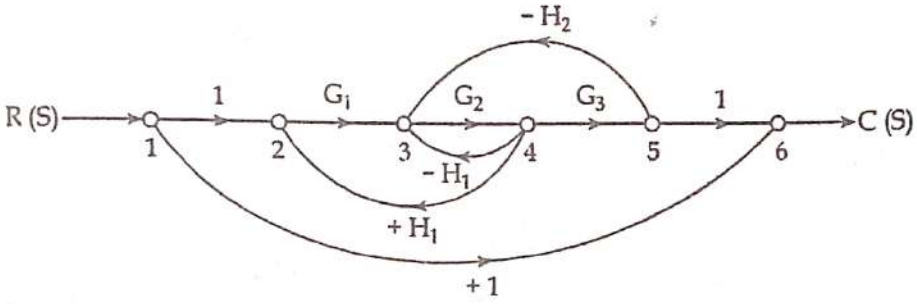
$\therefore$ $\boxed{T.F. = \frac{C(S)}{R(S)} = \frac{G_1G_2 - G_3}{1 + G_1H}}$

9. **Obtain the transfer function using signal flow graph.** **[2070 Ashadh]**



Solution:

Drawing signal flow graph;



Now, forward paths are;

$g_1 = 1 \times G_1 \times G_2 \times G_3 \times 1 = G_1G_2G_3$ $[1 \to 2 \to 3 \to 4 \to 5 \to 6]$

$g_2 = 1$ $[1 \to 6]$

and, individual loops are;

$L_1 = G_2 \times (-H_1) = -G_2H_1$ $[3 \to 4 \to 3]$

$L_2 = G_1 \times G_2 \times H_1 = G_1G_2H_1$ $[2 \to 3 \to 4 \to 2]$

$L_3 = G_2 \times G_3 \times (-H_2) = -G_2G_3H_2$ $[3 \to 4 \to 5 \to 3]$

$\Delta = 1 - [L_1 + L_2 + L_3] + [0]$

$= 1 - [-G_2H_1 + G_1G_2H_1 - G_2G_3H_2] = 1 + G_2H_1 - G_1G_2H_1 + G_2G_3H_2$

Again,

Δ_1 = Part of Δ not touching $g_1 = 1 - 0 = 1$

Δ_2 = Part of Δ not touching $g_2 = 1 - [L_1 + L_2 + L_3] = 1 + G_2H_1 - G_1G_2H_1 + G_2G_3H_2$

Transfer function (T.F.) $= \frac{C}{R} = \frac{\sum g_k \Delta_k}{\Delta} = \frac{g_1\Delta_1 + g_2\Delta_2}{\Delta}$

$\therefore$ $\boxed{T.F. = \frac{C}{R} = \frac{G_1G_2G_3 + [1 + G_2H_1 - G_1G_2H_1 + G_2G_3H_2]}{1 + G_2H_1 - G_1G_2H_1 + G_2G_3H_2}}$

10. **How do you characterize control systems in terms of speed, accuracy, stability and robustness?** **[2070 Ashadh, 2070 Chaitra]**

Ans: **i) In terms of speed**

Closed loop control systems are faster than open loop control system.

For detail

See the definition part 3.5.1

ii) In terms of accuracy

Closed loop control systems are more accurate than open loop system.

Scanned with CamScanner

For detail
See the definition part 3.5.6

iii) In terms of stability
The location of poles determines whether the system is stable or not.

For detail
See the definition part 3.5.4

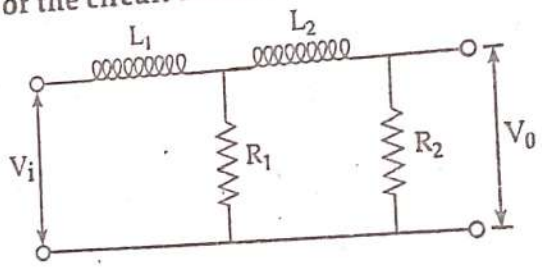
iv) In terms of robustness
Open loop control systems are directly affected by the external noise and disturbances; they show immediate change in output.
Closed loop control systems have better disturbance rejection capability. This is due to feedback. Increasing G(S)H(S) will suppress the noise and disturbances. Closed loop control systems are more accurate than open loop system as well.

For detail
See the definition part 3.5.5, 3.5.6 and 3.5.7

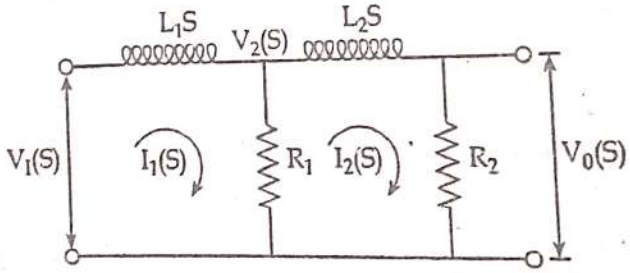
11. **Discuss how a feedback control system rejects the disturbance input.** [2070 Chaitra]

Ans: See the definition part 3.5.5, 3.5.6 and 3.5.7 [2070 Chaitra]

12. **Draw the block diagram of the circuit below.**



Solution:
The circuit in S-domain becomes;



Here,

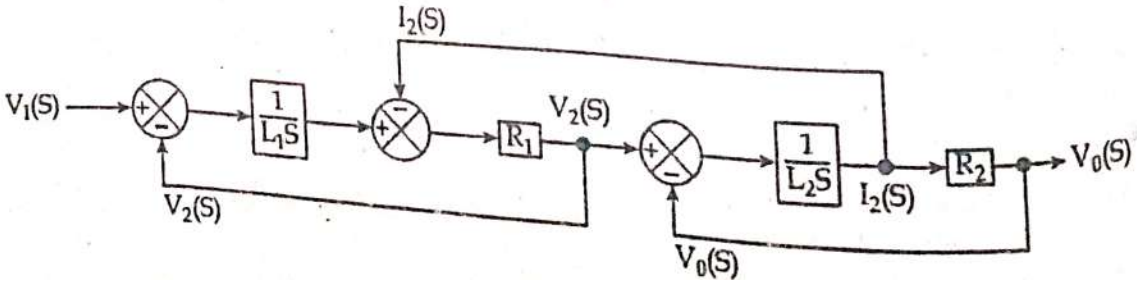
$$I_1(S) = \frac{V_1(S) - V_2(S)}{L_1 S}$$

$$V_2(S) = R_1[I_1(S) - I_2(S)]$$

$$I_2(S) = \frac{V_2(S) - V_0(S)}{L_2 S}$$

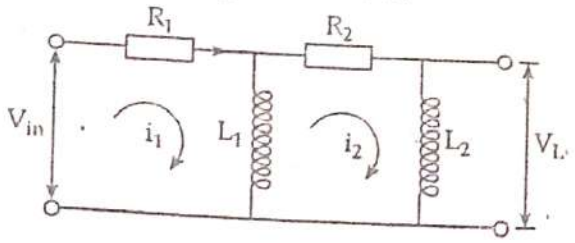
and, $V_0(S) = R_2 \cdot I_2(S)$

Now, drawing block diagram;



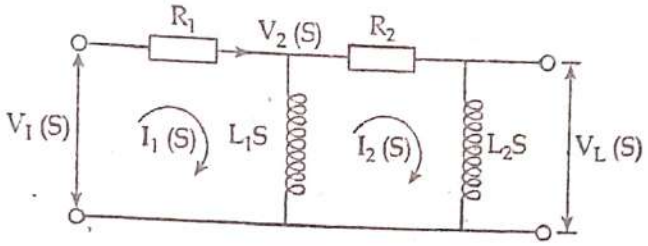
Scanned with CamScanner

13. Develop block diagram model for the system below. [2071 Chaitra]



Solution:

The circuit in S-domain becomes;



Here,

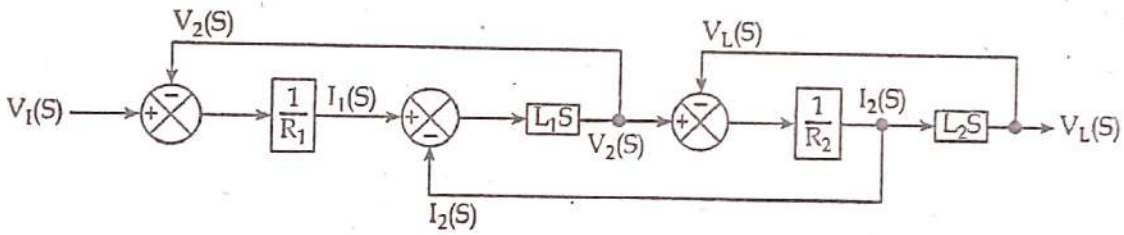
$$I_1(S) = \frac{V_1(S) - V_2(S)}{R_1} \quad (1)$$

$$V_2(S) = L_1(S)[I_1(S) - I_2(S)] \quad (2)$$

$$I_2(S) = \frac{1}{R_2}[V_2(S) - V_L(S)] \quad (3)$$

and, $V_L(S) = L_2(S) \cdot I_2(S)$ (4)

Now, drawing block diagram;



14. Effect of disturbance in case of feedback control system can be supposed by increasing the gain G(S) or H(S). [2071 Chaitra]

Ans: See the definition part 3.5.5 and 3.5.7

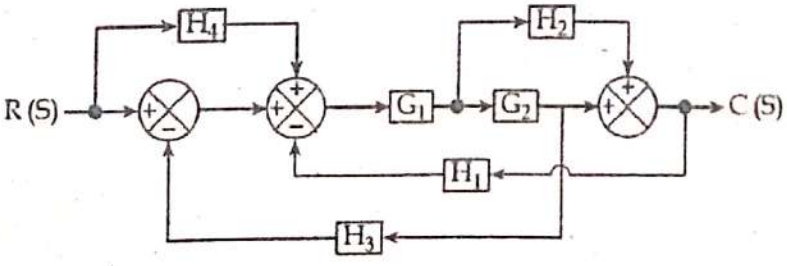
15. Discuss how dynamic responses of control system are affected by a feedback. [2071 Shrawan]

Ans: See the definition part 3.5.1

16. Explain why closed loop system has better disturbance rejection capability than open loop. [2072 Kartik]

Ans: See the definition part 3.5.5, 3.5.6 and 3.5.7

17. Determine the overall transfer functions $\frac{C(S)}{R(S)}$ of the given system by block diagram reduction technique. [2072 Chaitra]

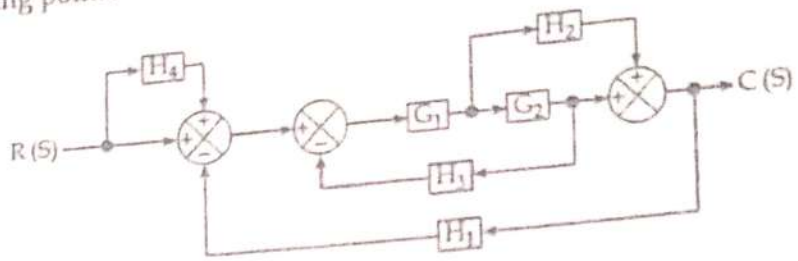


Solution:

Scanned with CamScanner

Interchanging summing points

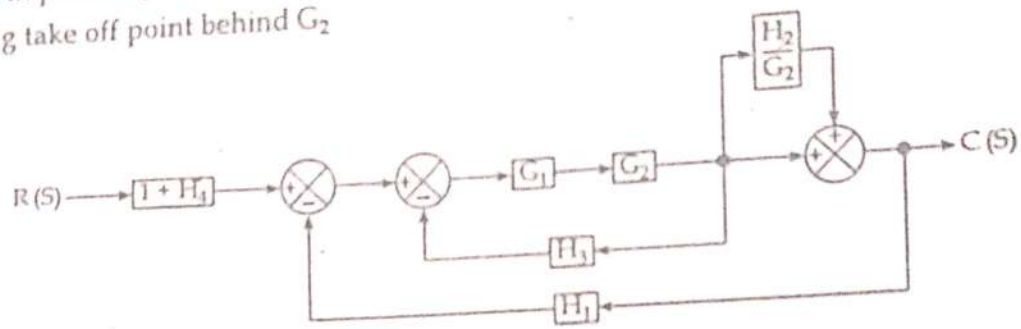
Step 1



Step 2

i) Blocks in parallel [H_4 with unity]

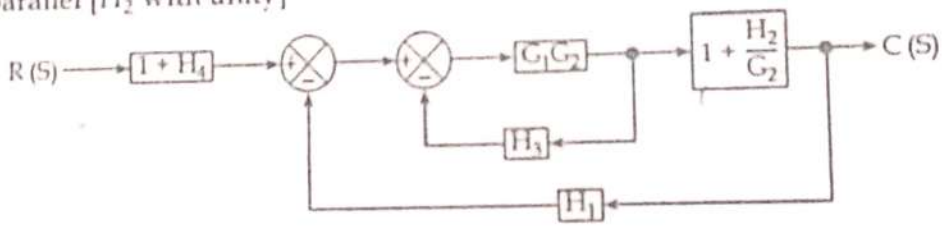
ii) Moving take off point behind G_2



Step 3

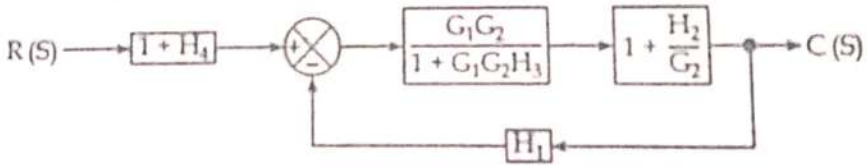
i) Blocks in cascade (G_1 and G_2)

ii) Block in parallel [H_2 with unity]



Step 4

Reducing negative feedback path of H_3



Here,

$$G(S) = G_1(S)G_2(S) = \frac{G_1G_2(G_2 + H_2)}{G_2(1 + G_1G_2H_3)} = \frac{G_1(G_2 + H_2)}{1 + G_1G_2H_3}$$

and, $H(S) = H_1$

$$\text{Transfer function} = \frac{G(S)}{1 + G(S)\,H(S)} = \frac{\dfrac{G_1(G_2 + H_2)}{1 + G_1G_2H_3}}{1 + \left[\dfrac{G_1(G_2 + H_2)}{1 + G_1G_2H_3}\right]H_1}$$

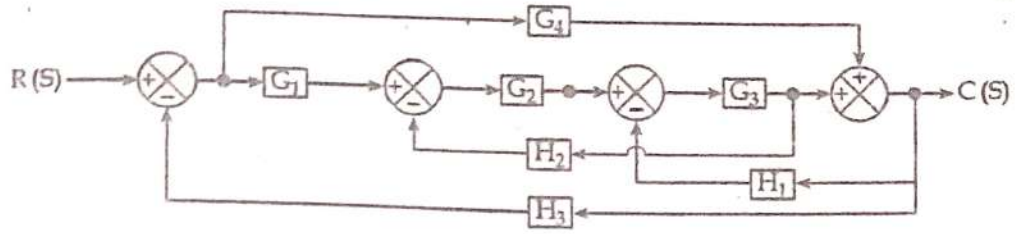
$$\therefore \quad \boxed{\text{T.F.} = \frac{C(S)}{R(S)} = \frac{G_1(G_2 + H_2)}{1 + G_1G_2H_3 + G_1(G_2 + H_2)H_1}}$$

18. Discuss how a closed loop system has better disturbance rejection and command input tracking capabilities in comparison to an open loop system. [2072 Chaitra]

Ans: See the definition part 3.5.5, 3.5.6 and 3.5.7

Scanned with CamScanner

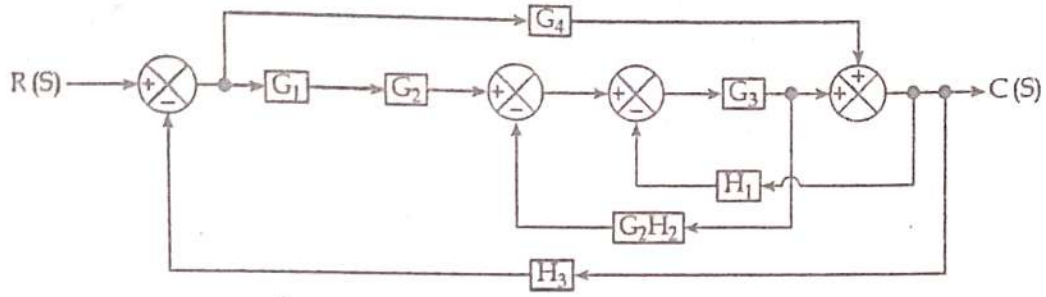
19. **Find the transfer function of the given system.** **[2073 Shrawan]**



Solution:

Step 1

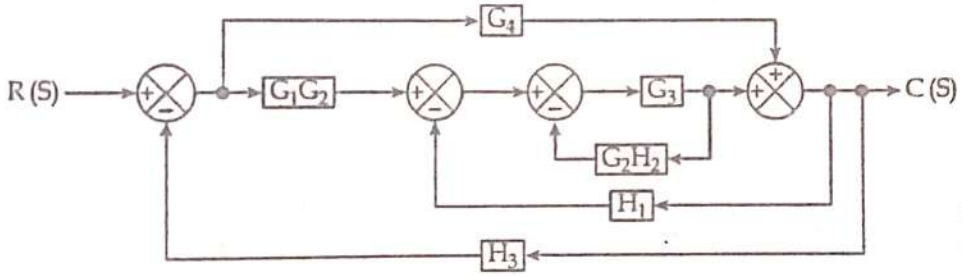
Moving summing point behind G_2:



Step 2

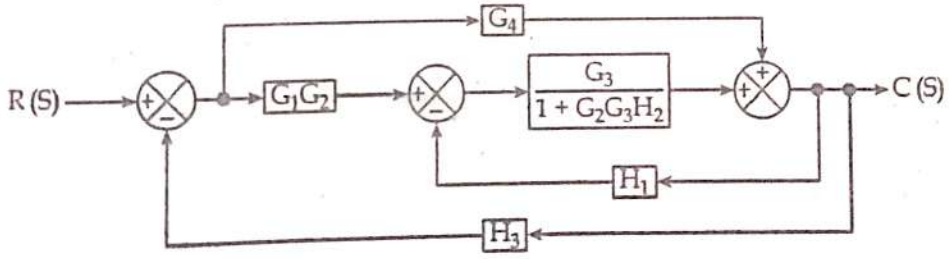
i) Block G_1 and G_2 are in cascade

ii) Exchanging summing points



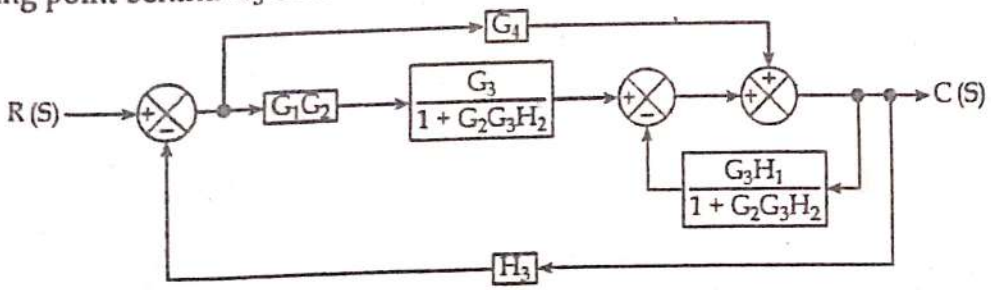
Step 3

Reducing negative feedback loop



Step 4

Moving summing point behind G_3 block:

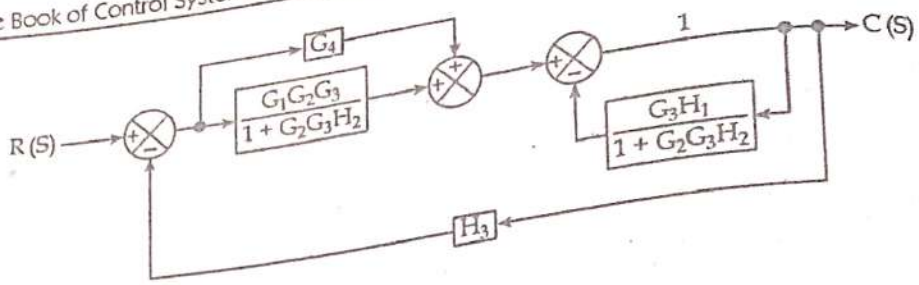


Step 5

i) Block in cascade

ii) Exchanging summing points

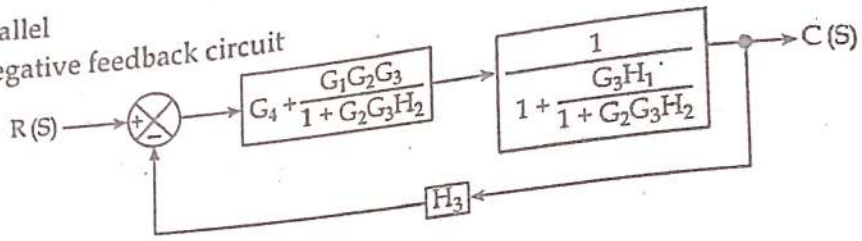
Scanned with CamScanner



Step 6

i) Block in parallel

ii) Reducing negative feedback circuit



Step 7

i) Blocks in cascade

ii) Reducing negative feedback circuit

$$G_1(S) = G_4 + \frac{G_1G_2G_3}{1+G_2G_3H_2} = \frac{G_1G_2G_3 + (1+G_2G_3H_2)}{(1+G_2G_3H_2)}$$

$$G_2(S) = \frac{(1+G_2G_3H_2)}{1+G_2G_3H_2+G_3H_1}$$

$$\text{and, } G(S) = G_1(S) \times G_2(S) = \frac{G_1G_2G_3 + G_4(1+G_2G_3H_2)}{1+G_2G_3H_2+G_3H_1}$$

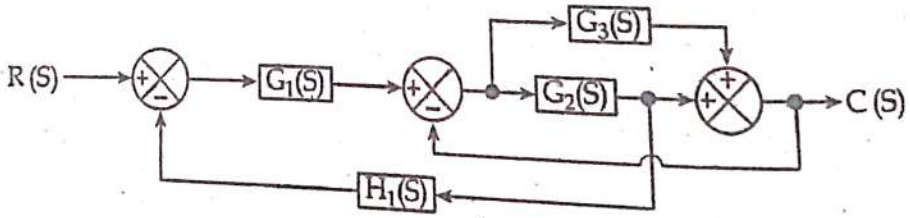
$$H(S) = H_3$$

so, reducing loops;

$$\text{T.F.} = \frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)} = \frac{\dfrac{G_1G_2G_3 + G_4(1+G_2G_3H_2)}{1+G_2G_3H_2+G_3H_1}}{1+\left[\dfrac{G_1G_2G_3 + G_4(1+G_2G_3H_2)}{1+G_2G_3H_2+G_3H_1}\right] \times H_3}$$

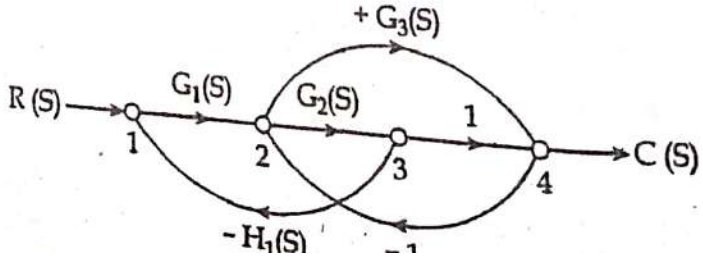
$$\therefore \boxed{\frac{C(S)}{R(S)} = \frac{G_1G_2G_3 + G_4(1+G_2G_3H_2)}{1+G_2G_3H_2+G_3H_1+[G_1G_2G_3 + G_4(1+G_2G_3H_2)]H_3}}$$

20. Using Masons gain formula. Find the transfer function $\frac{C(S)}{R(S)}$ of the system above. [2073 Chaitra]



Solution:

Drawing the S.F.G.;



Scanned with CamScanner

From the graph: we have,

Forward paths;

$g_1 = G_1(S) \times G_2(S) \times 1 = G_1(S) \cdot G_2(S)$ $[1 \to 2 \to 3 \to 4]$

$g_2 = G_1(S) \times G_3(S) = G_1(S) \cdot G_3(S)$ $[1 \to 2 \to 4]$

Individual loops;

$L_1 = G_1(S) \times G_2(S) \times [-H_1(S)] = -G_1(S)G_2(S)H_1(S)$ $[1 \to 2 \to 3 \to 1]$

$L_2 = G_2(S) \times 1 \times (-1) = -G_2(S)$ $[2 \to 3 \to 4 \to 2]$

$L_3 = G_3(S) \times (-1) = -G_3(S)$ $[2 \to 4 \to 2]$

and, $\Delta = 1 - [L_1 + L_2 + L_3] + 0$

$= 1 - [-G_1(S)G_2(S)H_1(S) - G_2(S) - G_3(S)]$

$= 1 + G_1(S)G_2(S)H_1(S) + G_2(S) + G_3(S)$

Δ_1 = Part of Δ not touching g_1 path $= 1 - 0 = 1$

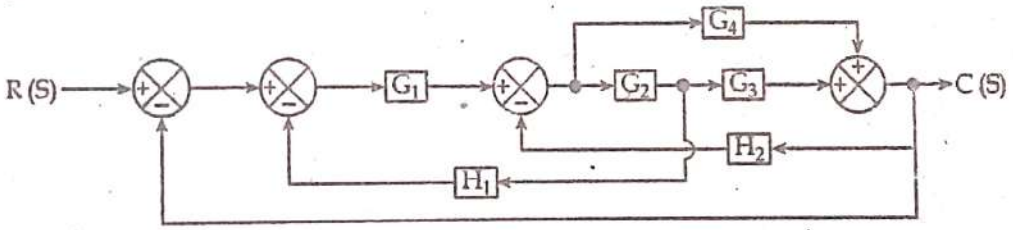
Δ_2 = Part of Δ not touching g_2 path $= 1 - 0 = 1$

From Masons gain formula; we have,

$$\text{T.F.} = \frac{C(S)}{R(S)} = \frac{\sum g_k \Delta_k}{\Delta} = \frac{g_1\Delta_1 + g_2\Delta_2}{\Delta} = \frac{G_1(S) \cdot G_2(S) + G_1(S) \cdot G_3(S)}{1 + G_2(S) + G_3(S) + G_1(S) \cdot G_2(S) \cdot H_1(S)}$$

$$\therefore \quad \boxed{\text{T.F.} = \frac{C(S)}{R(S)} = \frac{G_1(S)\,[G_2(S) + G_3(S)]}{1 + G_2(S) + G_3(S) + G_1(S) \cdot G_2(S) \cdot H_1(S)}}$$

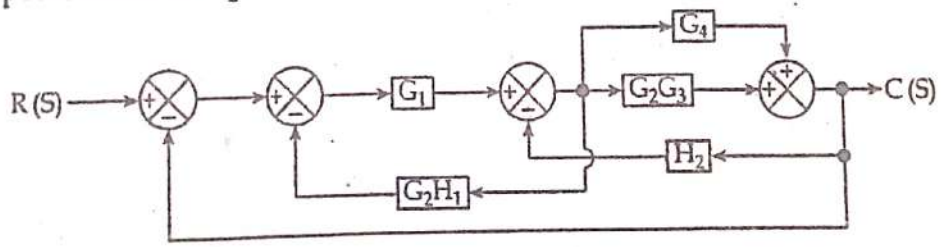
21. Determine the overall transfer function $\frac{C(S)}{R(S)}$ of the given system by block diagram reduction technique. **[2062 Poush]**



Solution:

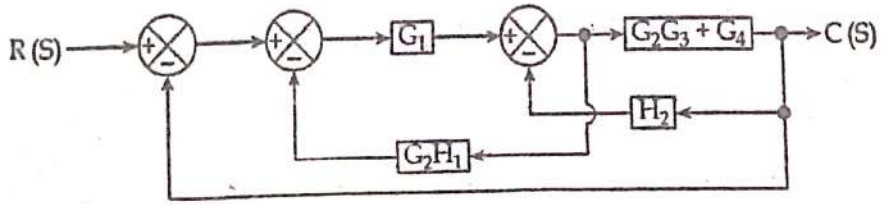
Step 1

Moving take off point ahead of G_2



Step 2

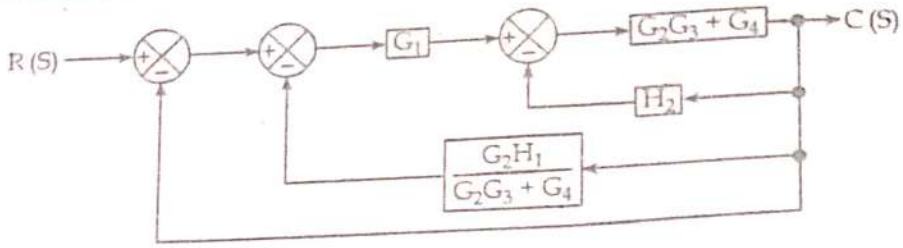
Blocks in parallel



Step 3

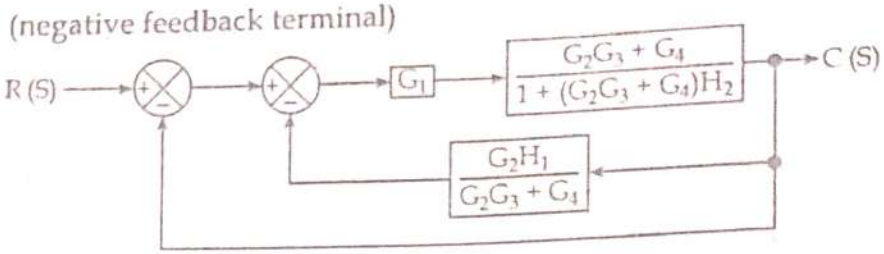
Moving take off point behind the block

Scanned with CamScanner



Step 4

Reducing inner loop (negative feedback terminal)



Step 5

Reducing negative feedback path

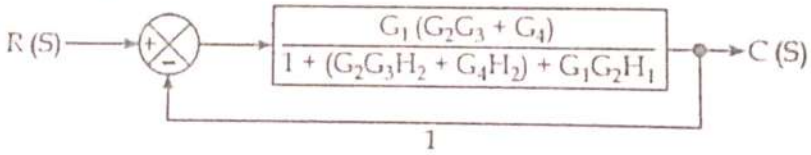
$$G(S) = G_1(S)G_2(S) = \frac{G_1(G_2G_3 + G_4)}{1 + (G_2G_3 + G_4)H_2}$$

and, $$H(S) = \frac{G_2H_1}{G_2G_3 + G_4}$$

so, $$\text{Transfer function} = \frac{G(S)}{1 + G(S)H(S)} = \frac{\dfrac{G_1(G_2G_3 + G_4)}{1 + (G_2G_3 + G_4)H_2}}{1 + \left[\dfrac{G_1(G_2G_3 + G_4)}{1 + (G_2G_3 + G_4)H_2}\right] \times \left(\dfrac{G_2H_1}{G_2G_3 + G_4}\right)}$$

$$= \frac{G_1(G_2G_3 + G_4)}{(1 + G_2G_3H_2 + G_4H_2) + G_1G_2H_1}$$

The block diagram now becomes;



Step 6

Reducing negative feedback path

Here,

$$H(S) = 1$$

so, $$\frac{C(S)}{R(S)} = \frac{\dfrac{G_1(G_2G_3 + G_4)}{(1 + G_2G_3H_2 + G_4H_2) + G_1G_2H_1}}{1 + \dfrac{G_1(G_2G_3 + G_4)}{(1 + G_2G_3H_2 + G_4H_2) + G_1G_2H_1}} = \frac{G_1(G_2G_3 + G_4)}{1 + G_2G_3H_2 + G_4H_2 + G_1G_2H_1 + G_1(G_2G_3 + G_4)}$$

$$\therefore \quad \boxed{\text{Transfer function} = \frac{C(S)}{R(S)} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_2G_3H_2 + G_1G_2H_1 + G_1G_2G_3 + G_1G_4 + G_4H_2}}$$

Scanned with CamScanner

Chapter 4

STABILITY

4.1 Introduction of stability and causes of instability 103
4.1.1 Introduction 103
4.1.2 Causes of instability 104
4.2 Characteristics equation, root location and stability 104
4.2.1 Characteristics equation 104
4.2.2 Root location and stability 104
4.3 Routh–Hurwitz stability criterion 107
4.3.1 Necessary but not sufficient condition for stability 107
4.3.2 R–H criteria process 107
4.3.3 Special cases of R–H stability criteria 110
4.4 Setting loop gain (K) using R–H criteria 118
4.5 Relative stability from complex plane axis shifting 121
4.5.1 Relative and absolute stability 121
4.5.2 Relative stability form complex plane shifting 122

4.1 INTRODUCTION OF STABILITY AND CAUSES OF INSTABILITY

4.1.1 Introduction

A system is said to be stable if its response **cannot** be made to increase indefinitely by the application of bounded input excitation. *i.e.,* If its output approaches towards infinite value for sufficiently large time, system is said to be unstable.

i.e., For bounded input excitation if;

i) $\lim_{t \to \infty} C(t) = 0$; the system is stable.

ii) $\lim_{t \to \infty} C(t) = \infty$; the system is unstable.

iii) $\lim_{t \to \infty} C(t) = K$; the system is marginally stable.

A linear time invariant (LTI) system is said to be stable if;

i) System is excited by bounded input, output is bounded. (BIBO stability criteria)

ii) In the absence of input, output tends towards zero. (The equilibrium state of system *i.e.,* steady state). This is known as **asymptotic stable.**

Scanned with CamScanner

A LTI system is said to be unstable if;

i) For a bounded input, it produces unbounded output.

ii) In absence of the input, output may not return to zero. It shows certain output without input.

4.1.2 Causes of Instability

i) If a system contains many dynamic variables and many feedback path, it introduces significa delay in the system which makes the system less stable (instable).

In a closed loop system giving negative feedback;

$$\text{Transfer function (T.F.)} = \frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)H(S)}$$

where, $1 + G(S)H(S) = 0$ represents the characteristics equation which on solution gives certai roots.

ii) If the roots have real part greater than zero *i.e.*, root lies on the R.H.P. (right half plane) of S plane, the system is unstable.

iii) If multiple roots at any location of imaginary axis then the system is unstable.

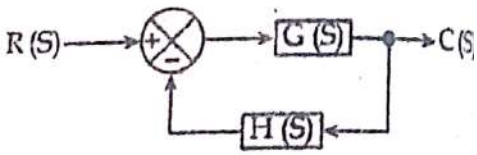
iv) If single pole at any location lies on imaginary axis, the system is marginally stable.

v) Sometimes positive feedbacks are cause of instability as they amplify unwanted changes in th system.

4.2 CHARACTERISTICS EQUATION, ROOT LOCATION AND STABILITY

4.2.1 Characteristics Equation

Transfer function of the above closed system is;



$$\text{Transfer function (T.F.)} = \frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)H(S)}$$

When the denominator term is equated with zero then, it is called characteristics equation.

i.e., $1 + G(S)H(S) = 0$; which is the characteristics equation.

The characteristics equation generally takes the form of;

$$a_0S^n + a_1S^{n-1} + a_2S^{n-2} + a_3S^{n-3} + \ldots\ldots + a_{n-1}S + a_n = 0$$

The solution gives some value of S and is termed as roots of equation which are commonly known as poles of control system.

4.2.2 Root Location and Stability

Poles on a graph are written by symbol ×.

Zero on a graph are written by symbol 0.

Graph is plotted as: σ (Sigma) + jω (Imaginary).

↓ Real axis ↓ Imaginary axis

) **Poles on negative real axis;**

Here,

$$G(S) = \frac{K}{S + a}$$

Pole lies at $S = -a$.

Taking inverse Laplace transform; we have,

$$g(t) = L^{-1}[G(S)] = L^{-1}\left[\frac{K}{S + a}\right] = Ke^{-at}$$

When $t = \infty$;

$g(t) = 0$

The system is stable.

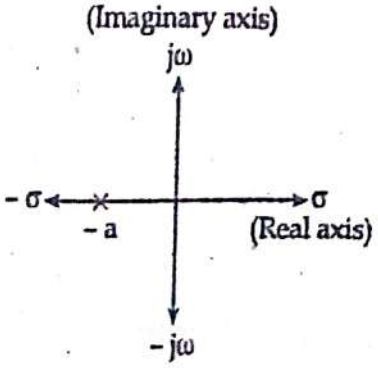


Figure: (i)

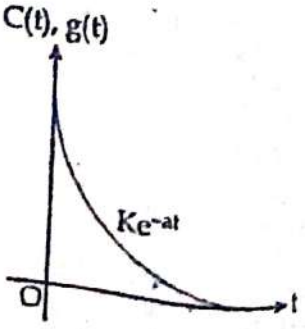


Figure: (ii)

Scanned with CamScanner

ii) **Pole on positive real axis:**

$$G(S) = \frac{K}{S-a}$$

Pole at $S = a$

Taking inverse Laplace transform; we have,

$$g(t) = L^{-1}[G(S)] = L^{-1}\left[\frac{K}{S-a}\right] = Ke^{at}$$

When $t = \infty$;

$g(t) = \infty$

The system is unstable.

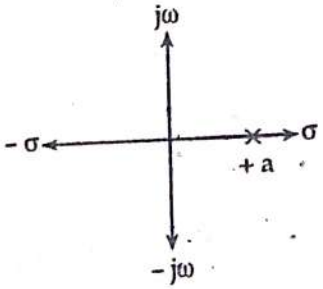


Figure: (i)

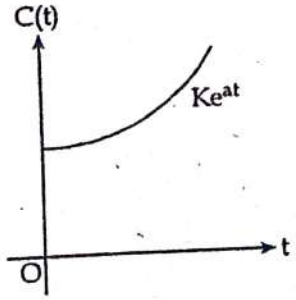


Figure: (ii)

iii) **Pole at origin**

$$G(S) = \frac{K}{S}$$

Pole at $S = 0$

Taking inverse Laplace transform; we have,

$$g(t) = L^{-1}[G(S)] = L^{-1}\left[\frac{K}{S}\right] = K$$

When $t = \infty$;

$g(t) = K$

Hence, the system is marginally stable.

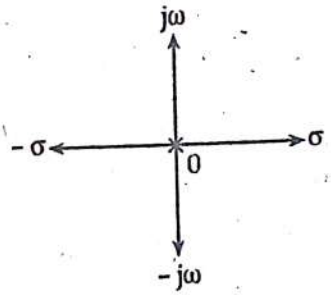


Figure: (i)

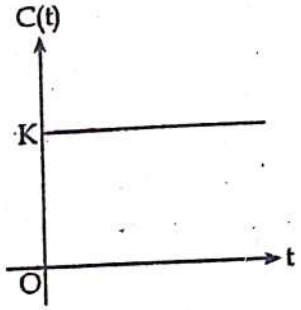


Figure: (ii)

iv) **Multiple pole at origin**

$$G(S) = \frac{K}{S^2}$$

Multiple (double) poles at $S = 0$

Taking inverse Laplace transform; we have,

$$g(t) = L^{-1}[G(S)] = L^{-1}\left[\frac{K}{S^2}\right] = Kt$$

When $t = \infty$;

$g(t) = \infty$

The system is unstable.

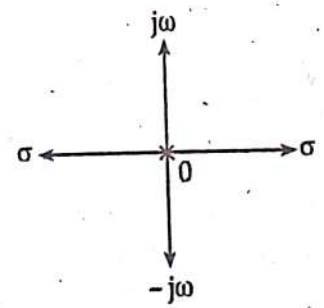


Figure: (i)

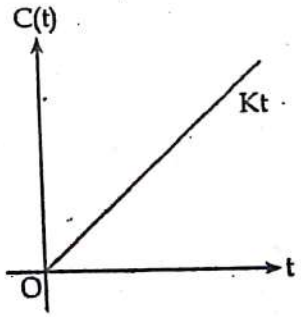


Figure: (ii)

v) **Complex pole in left half plane (L.H.P.)**

Let, complex conjugate pole $S = -\alpha \pm j\omega$ exists in a system as;

$$G(S) = \frac{K}{S+(\alpha+j\omega)} + \frac{K}{S+(\alpha-j\omega)}$$

Taking inverse Laplace; we have,

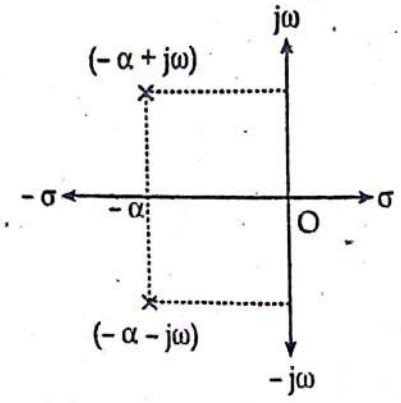


Figure: (i)

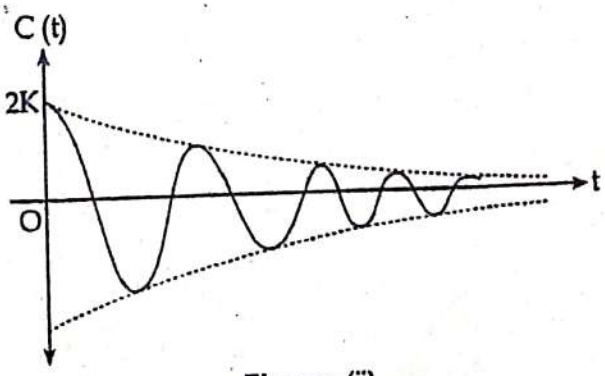


Figure: (ii)

Scanned with CamScanner

$$g(t) = L^{-1}\left[\frac{K}{S+(\alpha+j\omega)} + \frac{K}{S+(\alpha-j\omega)}\right] = L^{-1}\left[\frac{2K(S+\alpha)}{(S+\alpha)^2+\omega^2}\right] = 2Ke^{-\alpha t}\cos\omega t$$

When $t = \infty$;

$g(t) = 0$

The system is stable.

vi) **Complex pole in right half plane (R.H.P.)**

$S = \alpha \pm j\omega$ is complex pole in the right half plane.

$$G(S) = \frac{K}{S-\alpha+j\omega} + \frac{K}{S-\alpha-j\omega}$$

Taking inverse Laplace; we have,

$$g(t) = L^{-1}\left[\frac{K}{S-\alpha+j\omega} + \frac{K}{S-\alpha-j\omega}\right] = L^{-1}\left[\frac{2K(S-\alpha)}{(S-\alpha)^2+\omega^2}\right] = 2Ke^{\alpha t}\cos\omega t$$

When $t = \infty$;

Response $g(t) = \infty$

The system is unstable.

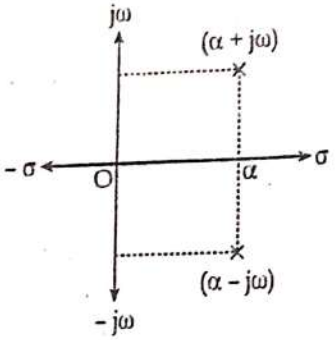


Figure: (i)

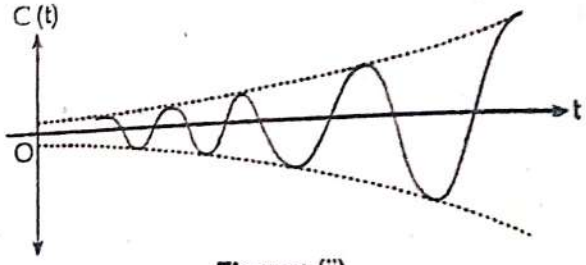


Figure: (ii)

vii) **Poles on imaginary (jω) axis**

Let, poles at $S = \pm j\omega$

$$G(S) = \frac{K}{S+j\omega} + \frac{K}{S-j\omega}$$

Taking inverse Laplace; we have,

$$g(t) = L^{-1}\left[\frac{K}{S+j\omega} + \frac{K}{S-j\omega}\right] = L^{-1}\left[\frac{2KS}{S^2+\omega^2}\right] = 2K\cos\omega t$$

When $t = \infty$;

$g(t) = \infty$

Hence, the system is unstable.

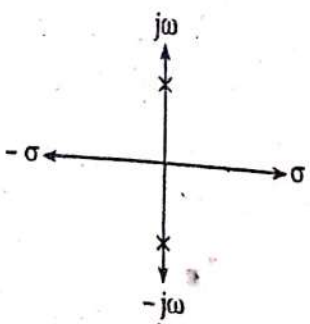


Figure: (i)

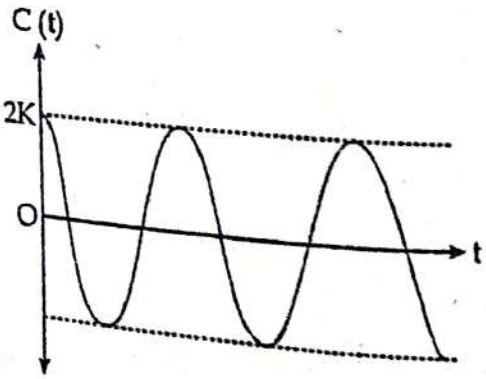


Figure: (ii)

Scanned with CamScanner

Summary

The necessary and sufficient condition that a feedback system be stable is that all the roots of characteristics equation $[1 + G(S)H(S) = 0]$ have negative real part of or all the poles should have negative real part.

i) Roots in L.H.P. → Stable
ii) At least one root in R.H.P. → Unstable
iii) Multiple pole on imaginary axis → Unstable
iv) Single pole on imaginary axis → Marginally stable

4.3 ROUTH–HURWITZ STABILITY CRITERION

The characteristics equation is in the form of;

$$a_0S^n + a_1S^{n-1} + a_2S^{n-2} + a_3S^{n-3} + \ldots\ldots + a_{n-1}S + a_n = 0$$

4.3.1 Necessary but Not Sufficient Condition for Stability

i) All the coefficient of characteristics equation should have same sign.

i.e., $a_0, a_1, a_2, a_3, \ldots\ldots, a_n$ should have same sign

ii) There should be no missing term.

If above two conditions are not satisfied the system will be unstable but if the coefficient have same sign and there is no missing term, we have no guarantee that the system will be stable. For stability, we use R-H criteria.

4.3.2 R–H Criteria Process

Step 1

Write characteristics equation in the decreasing order

$$a_0S^n + a_1S^{n-1} + a_2S^{n-2} + a_3S^{n-3} + \ldots\ldots + a_{n-1}S + a_n = 0$$

Step 2

Arrange the coefficients of given characteristics polynomial in two rows. The routh array will be;

First row	S^n	a_0	a_2	a_4	.....
Second row	S^{n-1}	a_1	a_3	a_5	.....

Step 3

Calculate elements of third row with the help of previous two rows by using the following formulae;

First row	S^n	a_0	a_2	a_4	a_6	.....
Second row	S^{n-1}	a_1	a_3	a_5	a_7	.....
Third row	S^{n-2}	b_1	b_2	b_3	b_4	.....

where, $b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}$

$$b_2 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}$$

$$b_3 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_6 \\ a_1 & a_7 \end{vmatrix}$$

$$b_4 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_8 \\ a_1 & a_9 \end{vmatrix}$$ and, so on.

Step 4

Calculate elements of fourth row with the help of previous two rows (just above) with the following formulae;

Scanned with CamScanner

First row	S^n	a_0	a_2	a_4	a_6	.....
Second row	S^{n-1}	a_1	a_3	a_5	a_7	.....
Third row	S^{n-2}	b_1	b_2	b_3	b_4	.....
Fourth row	S^{n-3}	C_1	C_2	C_3	C_4	.....
⋮	⋮	⋮	⋮	⋮	⋮	

and, so on up to S^0 row.

where, $C_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}$

$C_2 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}$

$C_3 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_7 \\ b_1 & b_4 \end{vmatrix}$

$C_4 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_9 \\ b_1 & b_5 \end{vmatrix}$ and, so on.

Step 5

Calculate the elements of remaining rows in the similar manner up to S^0 rows.

S^n	a_0	a_2	a_4	a_6	.....
S^{n-1}	a_1	a_3	a_5	a_7	.....
S^{n-2}	b_1	b_2	b_3	b_4	.....
S^{n-3}	C_1	C_2	C_3	C_4	.....
⋮	⋮	⋮	⋮	⋮	
S_0					

Concept

R-H criteria states that;

i) If same sign in first column, system is stable

ii) If sign changes, there are at least two roots in R.H.P.

iii) Total number of sign changes on first column is equal to the total number of roots lying on t right of S-plane.

EXAMPLE 4.1

Check whether the system is stable or NOT. The characteristics equation of given is;

$S^4 + 2S^3 + 6S^2 + 4S + 1 = 0$

Solution:

Step 1

All the coefficient have positive sign and no missing term; so, forming R-H array.

S^4	1	6	1
S^3	2	4	0

Step 2

S^4	1	6	1
S^3	2	4	0
S^2	b_1	b_2	

Scanned with CamScanner

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} 1 & 6 \\ 2 & 4 \end{vmatrix} = -\left(\frac{4-12}{2}\right) = 4$$

$$b_2 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -\frac{1}{2} \times (-2) = 1$$

Step 3

S^4	1	6	1
S^3	2	4	0
S^2	4	1	
S^1	C_1	C_2	

$$C_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} = -\frac{1}{4}\begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} = -\frac{1}{4}(2-16) = \frac{14}{4} = 3.5$$

$$C_2 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix} = -\frac{1}{4}\begin{vmatrix} 2 & 0 \\ 4 & 0 \end{vmatrix} = 0$$

Step 4

S^4	1	6	1
S^3	2	4	0
S^2	4	1	
S^1	3.5	0	
S^0	d_1		

$$d_1 = -\frac{1}{3.5}\begin{vmatrix} 4 & 1 \\ 3.5 & 0 \end{vmatrix} = -(-1) = 1$$

Step 5

The complete R-H array becomes;

S^4	1	6	1
S^3	2	4	0
S^2	4	1	
S^1	3.5	0	
S^0	1 ← Constant term of characteristics equation.		

Since there is no sign change in first column, so there is no roots in right half S-plane. So, the system is stable.

EXAMPLE 4.2

Determine the stability of:

$$\frac{C(S)}{R(S)} = \frac{S^2 + S + 1}{S^4 + 8S^3 + 18S^2 + 16S + 50}$$

[2070 Chaitra]

Solution:

Step 1

The characteristics equation of above transfer function is;

$$S^4 + 8S^3 + 18S^2 + 16S + 50 = 0$$

and, there is no missing term. All the coefficients have same positive sign.

Step 2

R-H array becomes;

Scanned with CamScanner

S^4 1 18 50

S^3 8 16 0

S^2 b_1 b_2

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = -\frac{1}{8}\begin{vmatrix} 1 & 18 \\ 8 & 16 \end{vmatrix} = -\left[\frac{16-(8\times 18)}{8}\right] = 16$$

$$b_2 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = -\frac{1}{8}\begin{vmatrix} 1 & 50 \\ 8 & 0 \end{vmatrix} = -\frac{1}{8}\times(-8)\times 50 = 50$$

Step 3

S^4 1 18 50

S^3 8 16 0

S^2 16 50

S^1 C_1 C_2

$$C_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} = -\frac{1}{16}\begin{vmatrix} 8 & 16 \\ 16 & 50 \end{vmatrix} = -\frac{1}{16}(8\times 50 - 16\times 16) = -9$$

$$C_2 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix} = -\frac{1}{16}\begin{vmatrix} 8 & 0 \\ 16 & 0 \end{vmatrix} = 0$$

Step 4

S^4 1 18 50

S^3 8 16 0

S^2 16 50

S^1 −9 0

S^0 d_1

$$d_1 = -\frac{1}{-9}\begin{vmatrix} 16 & 50 \\ -9 & 0 \end{vmatrix} = \frac{1}{9}(9\times 50) = 50$$

Step 5

The complete R–H array becomes;

S^4	1	18	50
S^3	8	16	0
S^2	16	50	
S^1	−9	0	
S^0	50	← Constant term of characteristics equation	

Since there are two sign changes [from 16 to −9 and −9 to 50] so there are two roots in right half S-plane. Therefore, the system is unstable.

4.3.3 Special Cases of R–H Stability Criteria

Case I

If the first column term in any row is zero but the remaining terms are not

EXAMPLE 4.3

Check the stability of system with characteristics equation $S^5 + S^4 + 2S^3 + 2S^2 + 3S + 5 = 0$.

Solution:

Step 1

No missing term and all the coefficient have same positive sign;

Scanned with CamScanner

S^5 1 2 3

S^4 1 2 5

S^3 b_1 b_2

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = -\frac{1}{1}\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = -(2-2) = 0$$

$$b_2 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = -\frac{1}{1}\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = -(5-3) = -2$$

Here, first column term is zero so the array becomes;

S^5 1 2 3

S^4 1 2 5

S^3 0 −2

We have two methods to solve these kinds of problem.

Method 1

Multiplying above characteristic equation by (S + a) such that 'a' is very small positive number.

For example;

Generally; we use (S + 1)

Now, solving; we get,

$$(S+1)(S^5 + S^4 + 2S^3 + 2S^2 + 3S + 5)$$

or, $S^6 + 2S^5 + 3S^4 + 4S^3 + 5S^2 + 8S + 5 = 0$

Step 2

Forming R-H array

S^6 1 3 5 5

S^5 2 4 8 0

S^6 b_1 b_2 b_3

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -\left(\frac{4-6}{2}\right) = 1$$

$$b_2 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} 1 & 5 \\ 2 & 8 \end{vmatrix} = -\left(\frac{8-10}{2}\right) = 1$$

$$b_3 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_6 \\ a_1 & a_7 \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} 1 & 5 \\ 2 & 0 \end{vmatrix} = -\left(\frac{-10}{2}\right) = 5$$

Step 3

S^6 1 3 5 5

S^5 2 4 8 0

S^4 1 1 5

S^3 C_1 C_2 C_3

$$C_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} = -\frac{1}{1}\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} = -(2-4) = 2$$

$$C_2 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix} = -\frac{1}{1}\begin{vmatrix} 2 & 8 \\ 1 & 5 \end{vmatrix} = -(10-8) = -2$$

$C_3 = 0$

Step 4

Array becomes;

Scanned with CamScanner

S^6	1	3	5	5
S^5	2	4	8	0
S^4	1	1	5	
S^3	2	−2	0	
S^2	d_1	d_2		

$$d_1 = -\frac{1}{2}\begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} = -\left(\frac{-2-2}{2}\right) = 2$$

$$d_2 = -\frac{1}{2}\begin{vmatrix} 1 & 5 \\ 2 & 0 \end{vmatrix} = -\left(\frac{-10}{2}\right) = 5$$

Step 5

The array becomes;

S^6	1	3	5	5
S^5	2	4	8	0
S^4	1	1	5	
S^3	2	−2	0	
S^2	2	5		
S^1	e_1	e_2		

$$e_1 = -\frac{1}{2}\begin{vmatrix} 2 & -2 \\ 2 & 5 \end{vmatrix} = -\left(\frac{10+4}{2}\right) = -7$$

$$e_2 = 0$$

The element that belongs to S^0 row is 5.

(*i.e.*, The constant term of the characteristics equation is always belongs to S^0 row.)

Step 6

The complete R-H array becomes;

S^6	1	3	5	5
S^5	2	4	8	0
S^4	1	1	5	
S^3	2	−2	0	
S^2	2	5		
S^1	−7	0		
S^0	5			

Since there are two sign changes in the first column of R-H table so there are two roots on right half S-plane. So, the system is unstable.

Method 2

Replace zero by small positive number or variable E;

Consider the following equation;

$$S^3 + S + 2 = 0$$

Step 1

Now the R-H array becomes;

$$S^3 + 0 \times S^2 + S + 2 = 0$$

Scanned with CamScanner

S^3	1	1
S^2	0	2
S^1	b_1	

$$b_1 = -\frac{1}{0}\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = \infty$$

Here, the zero on second row is replaced by a small positive number E.

Step 2

Therefore, the array becomes;

S^3	1	1
S^2	E	2
S^1	b_1	

$$b_1 = -\frac{1}{E}\begin{vmatrix} 1 & 1 \\ E & 2 \end{vmatrix} = -\left(\frac{2-E}{E}\right) = \frac{E-2}{E} = 1 - \frac{2}{E}$$

E is small positive number. Therefore, b_1 is negative.

Step 3

S^3	1	1
S^2	E	2
S^1	$\frac{E-2}{E}$	
S^0	2	

Number of sign changes in first column = 2

Number of roots in right half S-plane = 2

Therefore, the system is unstable.

Case II

When any row of R-H table is zero;

i) It indicates that the equation has at least one pair of roots which lie radically opposite to each other and equidistant from origin.

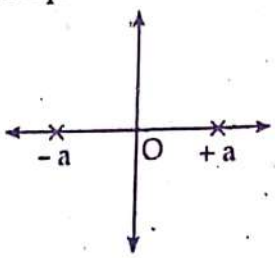


Figure: (i)

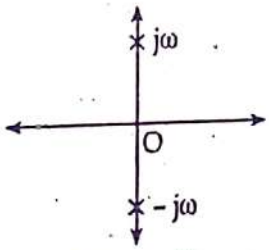


Figure: (ii)

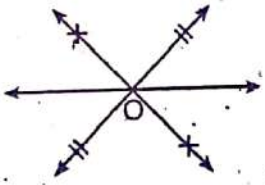


Figure: (iii)

ii) The solution for this kind of problem is solved by taking the auxiliary polynomial. The polynomial whose coefficients are the element of the row just above the row of zeros in R-H array is called an auxiliary polynomial.

EXAMPLE 4.4

Check the stability of $S^5 + 2S^4 + 24S^3 + 48S^2 - 25S - 50 = 0$. And find out the number of roots in right half plane.

Solution:

Step 1

The characteristics equation is;

Scanned with CamScanner

$$S^5 + 2S^4 + 24S^3 + 48S^2 - 25S - 50 = 0$$

No missing term, but the coefficient has different sign, so the system is **unstable. But, we have to find** out number of roots lying on R.H.P.

Step 1

S^5 1 24 −25

S^4 2 48 −50

S^3 b_1 b_2

$$b_1 = -\frac{1}{2}\begin{vmatrix} 1 & 24 \\ 2 & 48 \end{vmatrix} = -\left(\frac{48-48}{2}\right) = 0$$

$$b_2 = -\frac{1}{2}\begin{vmatrix} 1 & -25 \\ 2 & -50 \end{vmatrix} = -\left(\frac{-50+50}{2}\right) = 0$$

Here, all the elements of row-3 is zero. So, we need auxiliary equation.

Step 2

S^5 1 24 −25

S^4 2 48 −50

S^3 0 0

The auxiliary equation (A.E.) is that which is just above the row of zeros.

i.e., $A(S) = 2S^4 + 48S^2 - 50$

Taking derivative;

$$\frac{dA(S)}{dS} = 8S^3 + 96S$$

so, the array becomes;

Step 3

S^5 1 24 −25

S^4 2 48 −50

S^3 8 96

S^2 C_1 C_2

$$C_1 = -\frac{1}{8}\begin{vmatrix} 2 & 48 \\ 8 & 96 \end{vmatrix} = -\left(\frac{96\times 2 - 48\times 8}{8}\right) = 24$$

$$C_2 = -\frac{1}{8}\begin{vmatrix} 2 & -50 \\ 8 & 0 \end{vmatrix} = -\left(\frac{50\times 8}{8}\right) = -50$$

Step 4

S^5 1 24 −25

S^4 2 48 −50

S^3 8 96

S^2 24 −50

S^1 d_1

S^0 e_1

$$d_1 = -\frac{1}{24}\begin{vmatrix} 8 & 96 \\ 24 & -50 \end{vmatrix} = -\left[\frac{8\times(-50) - 96\times 24}{24}\right] = 112.667$$

and, e_1 = Constant term of characteristics equation = −50

Scanned with CamScanner

Stem 5

The complete R-H array;

S^5	1	24	−25
S^4	2	48	−50
S^3	8	96	
S^2	24	−50	
S^1	112.667		
S^0	−50		

Since, there is sign change in first column, the system is unstable.

Number of sign changes in first column = 1

Number of roots in R.H.P. = 1

EXAMPLE 4.5

Check the stability of $S^6 + 2S^5 + 8S^4 + 12S^3 + 20S^2 + 16S + 16 = 0$ [2071 Shrawan, 2073 Chaitra]

Solution:

Step 1

No missing term and all the coefficient have same sign.

The R-H array becomes;

S^6	1	8	20	16
S^5	2	12	16	0
S^4	b_1	b_2	b_3	

$$b_1 = -\frac{1}{2}\begin{vmatrix} 1 & 8 \\ 2 & 12 \end{vmatrix} = -\left(\frac{12-16}{2}\right) = 2$$

$$b_2 = -\frac{1}{2}\begin{vmatrix} 1 & 20 \\ 2 & 16 \end{vmatrix} = -\left(\frac{16-40}{2}\right) = 12$$

and, $$b_1 = -\frac{1}{2}\begin{vmatrix} 1 & 16 \\ 2 & 0 \end{vmatrix} = \left(\frac{32}{2}\right) = 16$$

Step 2

The R-H array;

S^6	1	8	20	16
S^5	2	12	16	0
S^4	2	12	16	
S^3	C_1	C_2	C_3	

$$C_1 = -\frac{1}{2}\begin{vmatrix} 2 & 12 \\ 2 & 12 \end{vmatrix} = 0$$

$$C_2 = -\frac{1}{2}\begin{vmatrix} 2 & 16 \\ 2 & 16 \end{vmatrix} = 0$$

and, $C_3 = 0$

Step 3

All the rows are zero, so we need auxiliary equation

Scanned with CamScanner

S^6	1	8	20	16
S^5	2	12	16	0
S^4	2	12	16	
S^3	0	0	0	

The auxiliary equation is;

$$A(S) = 2S^4 + 12S^2 + 16$$

Taking derivative; we have,

$$\frac{dA(S)}{dS} = 8S^3 + 24S$$

Step 4

The R-H array becomes;

S^6	1	8	20	16
S^5	2	12	16	0
S^4	2	12	16	
S^3	8	24		
S^2	d_1	d_2		

$$d_1 = -\frac{1}{8}\begin{vmatrix} 2 & 12 \\ 8 & 24 \end{vmatrix} = -\left(\frac{48 - 96}{8}\right) = 6$$

$$d_2 = -\frac{1}{8}\begin{vmatrix} 2 & 16 \\ 8 & 0 \end{vmatrix} = -\left(\frac{-8 \times 16}{8}\right) = 16$$

Step 5

The R-H array becomes;

S^6	1	8	20	16
S^5	2	12	16	0
S^4	2	12	16	
S^3	8	24		
S^2	6	16		
S^1	e_1			
S^0	f_1			

Here,

$$e_1 = -\frac{1}{6}\begin{vmatrix} 8 & 24 \\ 6 & 16 \end{vmatrix} = -\left(\frac{8 \times 16 - 24 \times 6}{6}\right) = \frac{8}{3}$$

and, f_1 = Constant term of characteristics equation = 16

Step 6

The complete R-H table is;

S^6	1	8	20	16
S^5	2	12	16	0
S^4	2	12	16	
S^3	8	24		
S^2	6	16		
S^1	2.67			
S^0	16			

Scanned with CamScanner

Since, no sign changes in the first column of R-H array so the system may be stable. So, we have to solve that auxiliary equation. Row of zero means they lie on imaginary axis.

Now, solving auxiliary equation; we have,

$$A(S) = 0$$

or, $2S^4 + 12S^2 + 16 = 0$

or, $S^4 + 6S^2 + 8 = 0$

Let, $x = S^2$; then,

$$x^2 + 6x + 8 = 0$$

or, $(x+2)(x+4) = 0$

Either, $x + 2 = 0$ or $x + 4 = 0$

$\therefore \quad x = -2$ $\therefore \quad x = -4$

and, $S^2 = -2$ and, $S^2 = -4$

or, $S^2 = (\pm j\sqrt{2})^2$ or, $S^2 = (\pm j2)^2$

$\therefore \quad S = \pm j\sqrt{2}$ $\therefore \quad S = \pm j2$

Let, the roots of auxiliary equations are the roots of characteristics equations as well.

$$S_1 = +j\sqrt{2}$$

$$S_2 = -j\sqrt{2}$$

$$S_3 = +j2$$

and, $S_4 = -j2$

The roots of auxiliary equations are always dominant roots *i.e.*, they lie close to imaginary axis, or they lie on imaginary axis or on the right half S-plane.

The remaining $(6 - 4 = 2)$ roots will always lie to the left half of S-plane and these two roots do not have significant role in stability because they lie far away from imaginary axis.

The roots of auxiliary equation are non-repeated on imaginary axis; hence, system is **marginally stable.**

Number of roots with zero real part = 4

Number of roots with positive real part = 0

Number of roots with negative real part = 2

If you are interested in finding out those tow roots then;

Divide the characteristics equation by auxiliary equation.

$$2S^4 + 12S^2 + 16 \;\overline{)\; S^6 + 2S^5 + 8S^4 + 12S^3 + 20S^2 + 16S + 16 \;(\; \tfrac{1}{2}S^2 + S + 1}$$

$$\underline{S^6 \quad \pm 6S^4 \quad \pm 8S^2}$$

$$2S^5 + 2S^4 + 12S^3 + 12S^2 + 16S + 16$$

$$\underline{2S^5 \quad \pm 12S^3 \quad \pm 16S}$$

$$2S^4 + 12S^2 + 16$$

$$\underline{2S^4 \pm 12S^2 \pm 16}$$

$$\times$$

The remaining two roots can be found by;

$$\frac{1}{2}S^2 + S + 1 = 0$$

or, $S^2 + 2S + 2 = 0$

or, $S = \dfrac{-2 \pm \sqrt{(2)^2 - 4 \times 1 \times 2}}{2} = \dfrac{-2 \pm \sqrt{(-4)}}{2} = \dfrac{-2 \pm j2}{2} = -1 \pm j$

Scanned with CamScanner

Therefore, the six roots of characteristics equations are;

$S_1 = +j\sqrt{2}$

$S_2 = -j\sqrt{2}$

$S_3 = +j2$

$S_4 = -j2$

$S_5 = -1 + j$

$S_6 = -1 - j$

4.4 SETTING LOOP GAIN (K) USING R–H CRITERIA

EXAMPLE 4.6

Consider the characteristics equation: $S^4 + 2S^3 + (K+4)S^2 + 9S + 25 = 0$. Using R–H criteria, determine the range of K for stability, value of K for marginal stability and frequency of undamped oscillation.

Solution:

Step 1

The characteristics equation is below and do not have missing term.

$$S^4 + 2S^3 + (K+4)S^2 + 9S + 25 = 0$$

R–H array becomes;

S^4	1	$(K+4)$	25
S^3	2	9	0
S^2	b_1	b_2	

$$b_1 = -\frac{1}{2}\begin{vmatrix} 1 & (K+4) \\ 2 & 9 \end{vmatrix} = -\left[\frac{9-2(K+4)}{2}\right] = -\left(\frac{9-2K-8}{2}\right) = \left(\frac{2K-1}{2}\right)$$

$$b_2 = -\frac{1}{2}\begin{vmatrix} 1 & 25 \\ 2 & 0 \end{vmatrix} = -\left(\frac{50}{2}\right) = 25$$

Step 2

The R–H array now becomes;

S^4	1	$(K+4)$	25
S^3	2	9	0
S^2	$\frac{2K-1}{2}$	25	
S^1	C_1	C_2	
S^0	d_1		

$$C_1 = -\frac{1}{\left(\frac{2K-1}{2}\right)}\begin{vmatrix} 2 & 9 \\ \frac{2K-1}{2} & 25 \end{vmatrix} = \left(\frac{2}{1-2K}\right)\left[50 - 9\times\left(\frac{2K-1}{2}\right)\right] = \frac{2}{1-2K}\times\frac{100-18K+9}{2}$$

$$= \frac{109-18K}{1-2K} = \frac{18K-109}{2K-1}$$

and, $C_2 = 0$

Also,

d_1 = Constant term of characteristics equation = 25

Scanned with CamScanner

Step 3

R-H array modifies to;

S^4	1	3	25
S^3	3	2	0
S^2	$\frac{7}{3}$	a	
S^1	$\frac{14-9a}{7}$	0	
S^0	a		

For stability;

$$\frac{14-9a}{7} > 0 \qquad \text{and, } a > 0$$

or, $14 - 9a > 0$

or, $14 > 9a$

or, $\frac{14}{9} > a$

∴ Range of stability is $a\left(0 < a < \frac{14}{9}\right)$

For marginal stability, any row must be equal to zero. Last row is not possible; so,

$$\frac{14-9a}{7} =$$

or, $14 = 9a$

or, $a = \frac{14}{9}$

∴ $\boxed{a = 1.56}$; for marginal stability

The auxiliary equation is;

$$A(S) = \left(\frac{7}{3}\right)S^2 + a = 0$$

or, $\left(\frac{7}{3}\right)S^2 = -\frac{14}{9}$

or, $S^2 = -\frac{2}{3}$

or, $S^2 = \left(\pm j\sqrt{\frac{2}{3}}\right)^2$

or, $S = \pm j\sqrt{\frac{2}{3}}$

and, $S = j\omega$

∴ $\boxed{\omega = +0.8165 \text{ rad/sec.}}$; which is the frequency of oscillation.

4.5 RELATIVE STABILITY FROM COMPLEX PLANE AXIS SHIFTING

4.5.1 Relative and Absolute Stability

Absolute stability means whether the system is stable or not. *i.e.*, The system is stable or unstable. If the system is stable then we determine how stable it is *i.e.*, we measure the degree of stability, it is known as relative stability.

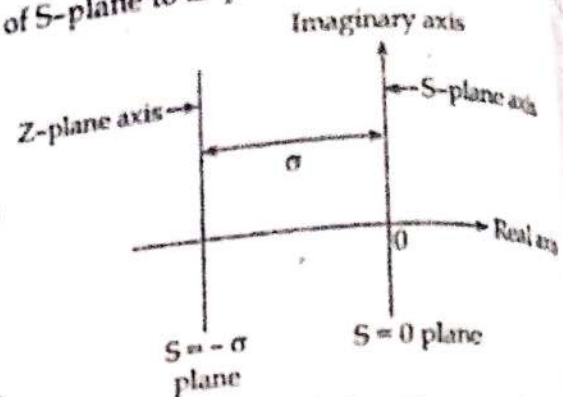
Scanned with CamScanner

In time domain the relative stability is measured by maximum overshoot and damping ratio. In frequency domain relative stability is measured by resonant peak (M_r).

4.5.2 Relative Stability Form Complex Plane Shifting

R-H criterion gives the information about the absolute stability. But we are more interested for the relative stability of the system. Relative stability can be examined by shifting the S-plane and then apply R-H stability criterion.

The characteristics equation is modified by shifting the origin of S-plane to Z-plane.



The reference axis is;

$$S = -\sigma$$

$$S + \sigma = 0$$

The reference axis is actually a Z-plane; so, $(Z = 0)$ axis means $(S + \sigma = 0)$ axis.

$$\therefore \quad Z = S + \sigma$$

or, $S = Z - \sigma$; in Z-plane

Now, applying R-H criteria, number of sign changes in the first column is equal to number roots to the right of the z-plane (*i.e.*, $S = -\sigma$ plane). In other words the roots of the original characteristics equation are more negative than $-\sigma$.

EXAMPLE 4.8

Check whether all the roots of characteristics equation $S^3 + 7S^2 + 25S + 39 = 0$ have real part more negative than -1.

Solution:

Step 1

Applying R-H criteria; we have,

S^3	1	25
S^2	7	39
S^1	b_1	
S^0	C_1	

Here,

$$b_1 = -\frac{1}{7}\begin{vmatrix} 1 & 25 \\ 7 & 39 \end{vmatrix} = -\left(\frac{39 - 25 \times 7}{7}\right) = 19.42$$

and, C_1 = Constant term = 39

Step 2

so, R-H table becomes;

S^3	1	25
S^2	7	39
S^1	19.42	
S^0	39	

Since, there is no sign change in first column; it means all the roots are in left half S-plane.

Step 3

Put, $S = Z - 1$

Then characteristics equation becomes;

Scanned with CamScanner

$$(Z-1)^3 + 7(Z-1)^2 + 25(Z-1) + 39 = 0$$

or, $$Z^3 - 1 - 3Z^2 + 3Z + 7(Z^2 - 2Z + 1) + 25Z - 25 + 39 = 0$$

or, $$Z^3 + 4Z^2 + 14Z + 20 = 0$$

Step 4

Applying R-H criteria; we have,

Z^3 1 14

Z^2 4 20

Z^1 b_1

Z^0 C_1

$$b_1 = -\frac{1}{4}\begin{vmatrix} 1 & 14 \\ 4 & 20 \end{vmatrix} = -\left(\frac{20 - 4 \times 14}{4}\right) = 9$$

and, C_1 = Constant term = 20

Step 5

R-H array completes into;

Z^3	1	14
Z^2	4	20
Z^1	9	
Z^0	20	

Since, no sign change in the first column, the roots of the characteristics equation lie in the left of Z-plane it means all the roots of S-domain lies on the left of S = −1 as well.

EXAMPLE 4.9

The open loop transfer function of a closed loop system is $G(S) = \frac{K(S+1)}{S(S+2)(S+3)}$, find maximum possible value of K for which the poles lie on the left of point (−0.5). [2070 Chaitra]

Solution:

Step 1

Here,

$$G(S) = \frac{K(S+1)}{S(S+2)(S+3)}$$

For a unity feedback system;

$$H(S) = 1$$

The characteristics equation is;

$$1 + G(S)\,H(S) = 0$$

or, $$1 + \frac{K(S+1)}{S(S+2)(S+3)} = 0$$

or, $$S(S+2)(S+3) + K(S+1) = 0$$

or, $$(S^2 + 2S)(S+3) + KS + K = 0$$

or, $$S^3 + 5S^2 + (6+K)S + K = 0$$

Put, $S = Z - 0.5$; then,

$$(Z-0.5)^3 + 5(Z-0.5)^2 + (6+K)(Z-0.5) + K = 0$$

or, $$Z^3 + 3.5Z^2 + (1.75 + K)Z + (0.5K - 1.875) = 0$$

Scanned with CamScanner

Step 2

Applying R-H criteria in Z-plane (S = −0.5)

Z^3 1 $(1.75 + K)$

Z^2 3.5 $(0.5K - 1.875)$

Z^1 b_1

Z^0 C_1

$$b_1 = -\frac{1}{3.5}\begin{vmatrix} 1 & 1.75 + K \\ 3.5 & 0.5K - 1.875 \end{vmatrix} = -\left[\frac{0.5K - 1.875 - 3.5(1.75 + K)}{3.5}\right] = -\left(\frac{-8 - 3K}{3.5}\right) = \left(\frac{8 + 3K}{3.5}\right)$$

and, C_1 = Constant term = $0.5K - 1.875$

Step 3

R-H array becomes;

Z^3	1	$(1.75 + K)$
Z^2	3.5	$(0.5K - 1.875)$
Z^1	$\left(\frac{8 + 3K}{3.5}\right)$	
Z^0	$(0.5K - 1.875)$	

For stability, the elements of first column should be positive; so,

$\frac{8 + 3K}{3.5} > 0$ and, $0.5K - 1.875 > 0$

or, $8 + 3K > 0$ or, $0.5K > 1.875$

or, $3K > -8$ $\therefore$ $K > 3.75$

$\therefore$ $K > -2.66$

$\therefore$ Range of K for stability is $K(3.75, \infty)$

Maximum possible value of K is infinite (*i.e.*, greater than 3.75)

Scanned with CamScanner

4.6 SOLVED PROBLEMS

Before solving questions, always remind this;

1. No zero in the first column, no any row zero and same sign on first column → Stable
2. No zero in the first column, no any row zero and sign changes on first column → Unstable
3. If any first column elements is zero and not others; then two methods;
   i) Multiply original characteristics equation by $(S+1)$ and do similarly. If sign changes in the first column then Unstable, if no sign changes the system is Stable.
   ii) Replace '0' by small positive number 'E'.
   If sign changes in first column → Unstable
   If sign doesn't change → Stable
4. If any row is zero, then you have to take derivative of auxiliary equation and continue.
   i) If sign changes in first column → Unstable
   ii) If no sign changes, you have to solve auxiliary equation
      1. If roots are non repeated → Marginally stable
      2. If roots are repeated in the imaginary axis → Unstable

1. Check whether the following characteristics equation has all its root in left of $S = -2$ or not?
$S^3 + 7S^2 + 26S + 39$. **[2061 Baishakh]**

Solution:

Step 1

Applying R–H criteria;

S^3	1	26
S^2	7	39
S^1	b_1	
S^0	C_1	

$$b_1 = -\frac{1}{7}\begin{vmatrix} 1 & 26 \\ 7 & 39 \end{vmatrix} = -\left(\frac{39 - 26 \times 7}{7}\right) = \frac{143}{7}$$

C_2 = Constant term of characteristics equation = 39

Step 2

The R–H array becomes;

S^3	1	26
S^2	7	39
S^1	$\frac{143}{7}$	
S^0	39	

All the term of first column is positive. So, all the roots lie on the left of $S = 0$ plane. (and the system is stable as well)

Step 3

Now to check at $S = -2$

Put, $S = Z - 2$ and characteristics equation becomes;

$$(Z-2)^3 + 7(Z-2)^2 + 26(Z-2) + 39 = 0$$

or, $Z^3 + Z^2 + 10Z + 7 = 0$

Scanned with CamScanner

Now, applying R-H criteria;

Z^3	1	10
Z^2	1	7
Z^1	d_1	
Z^0	e_1	

$$d_1 = -\frac{1}{1}\begin{vmatrix} 1 & 10 \\ 1 & 7 \end{vmatrix} = -(7-10) = 3$$

e_1 = Constant term = 7

Step 4

The R-H array becomes;

Z^3	1	10
Z^2	1	7
Z^1	3	
Z^0	7	

Since, all the elements of the first column have same sign; the system is also stable for Z-plane as well. This means that all the roots of original characteristics equation lies left from the $S = -2$ plane.

2. Determine the number of right half plane poles of the system, half plane poles of the system having characteristics equation as $S^5 + 7S^4 + 6S^3 + 42S^2 + 8S + 56 = 0$. [2067 Magh]

Solution:

Step 1

S^5	1	6	8
S^4	7	42	56
S^3	b_1	b_2	

$$b_1 = -\frac{1}{7}\begin{vmatrix} 1 & 6 \\ 7 & 42 \end{vmatrix} = -\left(\frac{42-42}{7}\right) = 0$$

$$b_2 = -\frac{1}{7}\begin{vmatrix} 1 & 8 \\ 7 & 56 \end{vmatrix} = -\left(\frac{56-56}{7}\right) = 0$$

All the row element are zeros.

Step 2

The auxiliary equation is;

$$A(S) = 7S^4 + 42S^2 + 56$$

Taking derivative;

$$\frac{dA(S)}{dS} = 28S^3 + 84S + 0 = 28S^3 + 84S$$

R-H array becomes;

S^5	1	6	8
S^4	7	42	56
S^3	28	84	
S^2	C_1	C_2	

$$C_1 = -\frac{1}{28}\begin{vmatrix} 7 & 42 \\ 28 & 84 \end{vmatrix} = -\left(\frac{7\times 84 - 42\times 28}{28}\right) = 21$$

Scanned with CamScanner

$$C_2 = -\frac{1}{28}\begin{vmatrix} 7 & 56 \\ 28 & 0 \end{vmatrix} = -\left(\frac{-28 \times 56}{28}\right) = 56$$

Step 3

R-H array becomes;

S^5	1	6	8
S^4	7	42	56
S^3	28	84	
S^2	21	56	
S^1	d_1		
S^0	e_1		

$$d_1 = -\frac{1}{21}\begin{vmatrix} 28 & 84 \\ 21 & 56 \end{vmatrix} = -\left(\frac{28 \times 56 - 21 \times 84}{21}\right) = \frac{28}{3}$$

and, e_1 = Constant term of characteristics equation = 56

Step 4

All the first column element have same sign; so, the system may be stable.

S^5	1	6	8
S^4	7	42	56
S^3	28	84	
S^2	21	56	
S^1	$\frac{28}{3}$		
S^0	56		

$$A(S) = 7S^4 + 42S^2 + 56 = 0$$

or, $7S^4 + 42S^2 + 56 = 0$

or, $S^4 + 6S^2 + 8 = 0$

Put, $x = S^2$; then,

$$x^2 + 6x + 8 = 0$$

or, $(x + 4)(x + 2) = 0$

or, $(S^2 + 4)(S^2 + 2) = 0$

Either, $S^2 + 4 = 0$ or, $S^2 + 2 = 0$

or, $S^2 = -4$ or, $S^2 = -2$

or, $S^2 = (\pm j2)^2$ or, $S^2 = (\pm j\sqrt{2})^2$

$\therefore$ $S = \pm j2$ $\therefore$ $S = \pm j\sqrt{2}$

Therefore, the roots of auxiliary equations are; $S_1 = +j$, $S_2 = -j$, $S_3 = +\sqrt{2}$ and $S_4 = -\sqrt{2}$.

The roots of auxiliary equation are the roots of characteristics equation as well. These roots lie nearer to imaginary axis and called dominant roots. The other root do not have significant role in stability and lies far in left half S-plane form imaginary axis.

If you are interested in finding out the roots;

Scanned with CamScanner

$$7S^4 + 42S^2 + 56 \overline{)\ S^5 + 7S^4 + 6S^3 + 42S^2 + 8S + 56\ (\ \frac{1}{7}S + 1}$$

$$\underline{-S^5 \quad \pm 6S^3 \quad \pm 8S}$$

$$7S^4 + 42S^2 + 56$$

$$\underline{-7S^4 \pm 42S^2 \pm 56}$$

$$\times$$

The other root can be;

$$\frac{1}{7}S + 1 = 0$$

or, $S + 7 = 0$

or, $S = -7$

∴ Remaining root $(S_5) = -7$

Step 5

Number of roots in R.H.P. = 0

Number of roots in L.H.P. = $(S_5) = 1$

Number of roots in imaginary axis = $(S_1, S_2, S_3, S_4) = 4$

The root lies on the imaginary axis and non-repeated. Therefore, the system is **marginally stable.**

3. The open loop transfer of a unity feedback control system is given is given by;

$$G(S) = \frac{K}{(S+2)(S+4)(S^2 + 6S + 25)}$$

Determine the range of gain K, for the system to be stable. Also find the value of K which will cause the sustained osculation and corresponding oscillation frequency. [2068 Baishakh]

Solution:

Step 1

We need characteristics equation;

$$G(S) = \frac{K}{(S+2)(S+4)(S^2 + 6S + 25)}$$

$$H(S) = 1$$

so, characteristics equation is;

$$1 + G(S)H(S) = 0$$

or, $(S+2)(S+4)(S^2 + 6S + 25) + K = 0$

or, $(S^2 + 6S + 8)(S^2 + 6S + 25) + K = 0$

or, $S^4 + 12S^3 + 69S^2 + 198S + (200 + K) = 0$

Step 2

Now, R-H array;

S^4	1	69	$(200 + K)$
S^3	12	198	0
S^2	b_1	b_2	
S^1	C_1		

$$b_1 = -\frac{1}{12}\begin{vmatrix} 1 & 69 \\ 12 & 198 \end{vmatrix} = -\left(\frac{198 - 69 \times 12}{10}\right) = \frac{105}{2}$$

$$b_2 = -\frac{1}{12}\begin{vmatrix} 1 & 200 + K \\ 12 & 0 \end{vmatrix} = -\left[\frac{-12(200 + K)}{12}\right] = 200 + K$$

Scanned with CamScanner

Step 3

R-H array;

S^4 1 69 $(200 + K)$

S^3 12 198 0

S^2 $\frac{105}{2}$ $200 + K$

S^1 C_1 C_2

S^0 d_1

$$C_1 = -\frac{1}{\frac{105}{2}}\begin{vmatrix} 12 & 198 \\ \frac{105}{2} & 200 + K \end{vmatrix} = -\frac{2}{105}\left[12(200 + K) - \left(198 \times \frac{105}{2}\right)\right] = -\frac{2}{105}[12K - 7995]$$

$$= \left(\frac{1066}{7} - \frac{8K}{35}\right) = \left(\frac{5330 - 8K}{35}\right)$$

$$C_2 = -\frac{1}{\frac{105}{2}}\begin{vmatrix} 12 & 0 \\ \frac{105}{2} & 0 \end{vmatrix} = 0$$

and, d_1 = Constant term of characteristics equation = $200 + K$

Step 4

R-H array completed into;

S^4 1 69 $(200 + K)$

S^3 12 198 0

S^2 $\left(\frac{105}{2}\right)$ $(200 + K)$

S^1 $\left(\frac{5330 - 8K}{35}\right)$

S^0 $(200 + K)$

The system will be stable when all elements of first column are positive.

so, $\frac{5330 - 8K}{35} > 0$ and, $200 + K > 0$

or, $5330 - 8K > 0$ or, $K > -200$

or, $5330 > 8K$

or, $\frac{2665}{4} > K$

$\therefore$ $K < 666.25$

$\therefore$ The range of K to be stable is $(-200 < K < 666.25)$.

Step 5

For marginal stability;

A row must contain all elements zero, which is only possible for S^1 row (since, last row must contain constant term, and never be zero.)

Hence, auxiliary equation is;

$$A(S) = \left(\frac{105}{2}\right)S^2 + (200 + K) = 0$$

Scanned with CamScanner

and, $\frac{5330 - 8K}{35} = 0$

or, $5330 - 8K = 0$

or, $\frac{5330}{8} = K$

$\therefore$ $K = 666.25$; which is the value for marginal stability.

The frequency of sustained oscillation can be found form auxiliary equation.

i.e., $A(S) = \left(\frac{105}{2}\right)S^2 + \left(200 + \frac{5330}{8}\right) = 0$

or, $\left(\frac{105}{2}\right)S^2 + \frac{3465}{4} = 0$

or, $\left(\frac{105}{2}\right)S^2 = -\frac{3465}{4}$

or, $S^2 = -\left(\frac{33}{2}\right)$

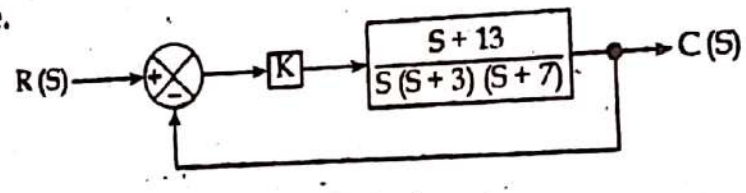
or, $S^2 = \left(\pm j\sqrt{\frac{33}{2}}\right)^2$

and, $S = j\omega$

so, ω = Frequency $= \left(\pm\sqrt{\frac{33}{2}}\right) = \pm 4.062$ rad/sec.

$\therefore$ Frequency of oscillation $(\omega) = \pm\sqrt{\frac{33}{2}} = 4.062$ rad/sec.

4. Apply R-H criteria to determine the range of K for a unity feedback system with following system to be stable. [2068 Chaitra]



Solution:

Step 1

We need characteristics equation:

Here,

$$G(S) = \frac{K(S+13)}{S(S+3)(S+7)}$$

$$H(S) = 1$$

and, characteristic equation is;

$$1 + G(S)H(S) = 0$$

or, $1 + \frac{K(S+13)}{S(S+3)(S+7)} = 0$

or, $S(S+3)(S+7) + K(S+13) = 0$

or, $(S^2 + 3S)(S+7) + KS + 13K = 0$

or, $S^3 + 10S^2 + (21 + K)S + 13K = 0$

Now, R-H array of characteristics equation;

S^3	1	$21 + K$
S^2	10	$13K$
S^1	b_1	
S^0	C_1	

Scanned with CamScanner

$$b_1 = -\frac{1}{10}\begin{vmatrix} 1 & 21+K \\ 10 & 13K \end{vmatrix} = -\left(\frac{13K - 10(21+K)}{10}\right) = -\left(\frac{3K - 210}{10}\right) = \frac{210 - 3K}{10}$$

and, C_1 = Constant term of characteristics equation = 13K

Step 2

The array now becomes;

S^3	1	21 + K
S^2	10	13K
S^1	$\frac{210 - 3K}{10}$	
S^0	13K	

The system will be stable only if first column term all have same sign;

i.e., $\frac{210 - 3K}{10} > 0$ and, $13K > 0$

or, $210 - 3K > 0$ or, $K > 0$

or, $70 > K$

∴ The range of K for stability is K(0, 70).

i.e., K should be $(0 < K < 70)$

5. **Check the stability of the system having characteristics equation.** **[2072 Kartik]**

$$S^5 + 4S^4 + 2S^3 + 8S^2 + S + 4 = 0$$

Solution:

Step 1

No missing terms and all the coefficient have same sign. So, the R-H array becomes;

$$S^5 + 4S^4 + 2S^3 + 8S^2 + S + 4 = 0$$

S^5	1	2	1
S^4	4	8	4
S^3	b_1	b_2	
S^2	C_1	C_2	
S^1	d_1		
S^0	e_1		

$$b_1 = -\frac{1}{4}\begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix} = -\left(\frac{8-8}{4}\right) = 0$$

$$b_2 = -\frac{1}{4}\begin{vmatrix} 1 & 1 \\ 4 & 4 \end{vmatrix} = 0$$

Here, S^3 row is zero; so,

Step 2

R-H array becomes;

S^5	1	2	1
S^4	4	8	4
S^3	0	0	

The auxiliary equation is;

Scanned with CamScanner

$$A(S) = 4S^4 + 8S^2 + 4$$

Taking derivative; we have,

$$\frac{dA(S)}{dS} = 16S^3 + 16S$$

Step 3

The R-H array now becomes;

S^5	1	2	1
S^4	4	8	4
S^3	16	16	
S^2	C_1	C_2	

$$C_1 = -\frac{1}{16}\begin{vmatrix} 4 & 8 \\ 16 & 16 \end{vmatrix} = -\left(\frac{64 - 128}{16}\right) = 4$$

$$C_2 = -\frac{1}{16}\begin{vmatrix} 4 & 4 \\ 16 & 0 \end{vmatrix} = -\left[\frac{(-4) \times 16}{16}\right] = 4$$

Step 4

R-H array becomes;

S^5	1	2	1
S^4	4	8	4
S^3	16	16	
S^2	4	4	
S^1	d_1		
S^0	e_1		

Now,

$$d_1 = -\frac{1}{4}\begin{vmatrix} 16 & 16 \\ 4 & 4 \end{vmatrix} = 0$$

Here, first column element is zero and row S^1 is also zero; so let us suppose auxiliary equation again;

So, the auxiliary equation is;

$$B(S) = 4S^2 + 4$$

Taking derivative; we have,

$$\frac{dB(S)}{dS} = 8S$$

Step 5

The R-H table now becomes;

S^5	1	2	1
S^4	4	8	4
S^3	16	16	
S^2	4	4	
S^1	8		
S^0	4		

The first column of the R-H, array has same sign. So, it may be a stable system.
To find out roots we need to solve auxiliary equations.

Scanned with CamScanner

Step 6

$$A(S) = 4S^4 + 8S^2 + 4$$

or, $S^4 + 2S^2 + 1 = 0$

or, $(S^2 + 1)(S^2 + 1) = 0$

The roots are;

$$S^2 = -1$$

or, $S^2 = (\pm j)^2$

$\therefore$ $S = \pm j$

The four roots are; $S_1 = +j$, $S_2 = -j$, $S_3 = +j$ and $S_4 = -j$.

and, solving other auxiliary equation;

$$B(S) = 4S^2 + 4 = 0$$

or, $4S^2 + 4 = 0$

or, $4S^2 = -4$

or, $S^2 = -1$

or, $S^2 = (\pm j)^2$

$\therefore$ $S = \pm j$

The remaining $(6 - 4 = 2)$ roots are $S_5 = +j$ and $S_6 = -j$.

All the six roots lie on a imaginary axis since the roots are repeated, the system is unstable.

6. **Using R-H criteria tell how many roots of polynomial equation is in right half s-plane, in left half-S-plane and on jω axis and also comment on stability. $S^6 + 3S^5 + 4S^4 + 6S^3 + 5S^2 + 3S + 2 = 0$** [2069 Chaitra, 2071 Chaitra, 2074 Ashwin]

Solution:

Step 1

R-H array;

S^6	1	4	5	2
S^5	3	6	3	0
S^4	b_1	b_2	b_3	

$$b_1 = -\frac{1}{3}\begin{vmatrix} 1 & 4 \\ 3 & 6 \end{vmatrix} = -\left(\frac{6-12}{3}\right) = 2$$

$$b_2 = -\frac{1}{3}\begin{vmatrix} 1 & 5 \\ 3 & 3 \end{vmatrix} = -\left(\frac{3-15}{3}\right) = 4$$

$$b_3 = -\frac{1}{3}\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} = \left(\frac{6}{3}\right) = 2$$

Step 2

R-H array modifies into;

S^6	1	4	5	2
S^5	3	6	3	0
S^4	2	4	2	
S^3	C_1	C_2	C_3	

Scanned with CamScanner

$$C_1 = -\frac{1}{2}\begin{vmatrix} 3 & 6 \\ 2 & 4 \end{vmatrix} = 0$$

$$C_2 = -\frac{1}{2}\begin{vmatrix} 3 & 6 \\ 2 & 4 \end{vmatrix} = 0$$

$$C_3 = 0$$

Since, all the rows are zero; we need to take the derivative of auxiliary equation.

$$A(S) = 2S^4 + 4S^2 + 2$$

Taking derivative; we have,

$$\frac{dA(S)}{dS} = 8S^3 + 8S$$

Step 3

Now, the array becomes;

S^6	1	4	5	2
S^5	3	6	3	0
S^4	2	4	2	
S^3	8	8	0	
S^2	d_1	d_2		

$$d_1 = -\frac{1}{8}\begin{vmatrix} 2 & 4 \\ 8 & 8 \end{vmatrix} = -\left(\frac{16-32}{8}\right) = 2$$

$$d_2 = -\frac{1}{8}\begin{vmatrix} 2 & 2 \\ 8 & 0 \end{vmatrix} = \left(\frac{16}{8}\right) = 2$$

Step 4

The table now;

S^6	1	4	5	2
S^5	3	6	3	0
S^4	2	4	2	
S^3	8	8	0	
S^2	2	2		
S^1	e_1	e_2		
S^0	f_1			

$$e_1 = -\frac{1}{2}\begin{vmatrix} 8 & 8 \\ 2 & 2 \end{vmatrix} = 0$$

$$e_2 = -\frac{1}{2}\begin{vmatrix} 8 & 0 \\ 2 & 0 \end{vmatrix} = 0$$

Since, all the rows are zero; we have to take derivation of auxiliary equation. The auxiliary equation

$$B(S) = 2S^2 + 2$$

Taking derivative; we have,

$$\frac{dA(S)}{dS} = 4S$$

and, f_1 = Constant term of characteristics equation = 2

Step 5

The final and complete R–H array is;

Scanned with CamScanner

S^6	1	4	5	2
S^5	3	6	3	0
S^4	2	4	2	
S^3	8	8	0	
S^2	2	2		
S^1	4	0		
S^0	2			

The first column terms have same sign and may be stable. To check the stability we have to find out roots of auxiliary equation. The roots of auxiliary equations are dominant roots and roots of characteristics equation as well.

Step 6

The auxiliary equation is;

$$A(S) = 2S^4 + 4S^2 + 2 = 0$$

or, $2S^4 + 4S^2 + 2 = 0$

Put, $x = S^2$

$$2x^2 + 4x + 2 = 0$$

or, $x^2 + 2x + 1 = 0$

or, $(x + 1)(x + 1) = 0$

or, $(S^2 + 1)(S^2 + 1) = 0$

The four roots are;

$$S^2 = -1$$

or, $S^2 = (\pm j)^2$

$\therefore$ $S = \pm j$

The roots are; $S_1 = +j$, $S_2 = -j$, $S_3 = +j$ and $S_4 = -j$.

The remaining $(6 - 4 = 2)$ roots can be found by;

$$B(S) = 2S^2 + 2 = 0$$

or, $2S^2 + 2 = 0$

or, $S^2 = -1$

or, $S^2 = (\pm j)^2$

$\therefore$ $S = \pm j$

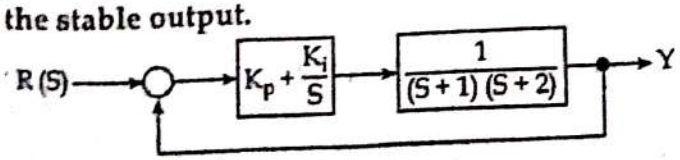
The remaining roots are; $S_5 = +j$ and $S_6 = -j$.

All the roots lie on imaginary axis. Since, the roots are repeated, the system is unstable.

Step 7 Solutions

i) System is unstable
ii) Number of roots in R.H.P. = 0
iii) Number of roots in L.H.P. = 0
iv) Number of roots in jω axis = 6 (All roots)

7. For a closed loop system given, determine the range of controller gain (K_p, K_i) so that the PI controller provides the stable output. [2073 Shrawan]



Solution:

Scanned with CamScanner

Here,

Forward transfer function $G(S) = \left(K_p + \frac{K_i}{S}\right) \times \frac{1}{(S+1)(S+2)}$

and, $H(S) = 1$

so, characteristics equation is;

$$1 + G(S)H(S) = 0$$

$$1 + \left(K_p + \frac{K_i}{S}\right) \times \frac{1}{(S+1)(S+2)} = 0$$

or, $$1 + \left(\frac{SK_p + K_i}{S}\right) \times \frac{1}{(S+1)(S+2)} = 0$$

or, $$S(S+1)(S+2) + (SK_p + K_i) = 0$$

or, $$S(S^2 + 3S + 2) + (SK_p + K_i) = 0$$

or, $$S^3 + 3S^2 + (2 + K_p)S + K_i = 0$$

Now, forming R-H array; we have,

S^3	1	$(2 + K_p)$
S^2	3	K_i
S^1	b_1	
S^0	c_1	

and, $b_1 = -\frac{1}{3}\begin{vmatrix} 1 & (2+K_p) \\ 3 & K_i \end{vmatrix} = -\frac{K_i - 6 - 3K_p}{3} = \frac{6 - 3K_p - K_i}{3}$

and, c_1 = Constant term of characteristics equation $= K_i$

so, R-H array becomes;

S^3	1	$(2 + K_p)$
S^2	2	K_i
S^1	$\left(\frac{6 - 3K_p - K_i}{3}\right)$	
S^0	K_i	

For stability, the first column of R-H array should have same positive sign.

so, $\left(\frac{6 - 3K_p - K_i}{3}\right) > 0$ and, $K_i > 0$

or, $(6 - 3K_p - K_i) > 0$

or, $(6 - 3K_p) > K_i$

and from characteristics equation, all the coefficients should have same sign.

so, $(2 + K_p) > 0$

or, $K_p > (-2)$

Hence, range of constant are $(6 - 3K_p) > K_i > 0$ and $K_p > (-2)$.

Scanned with CamScanner

Chapter 5

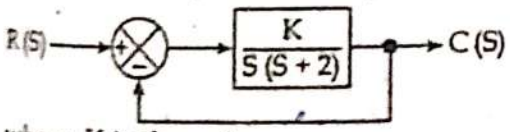
ROOT LOCUS TECHNIQUE

5.1	Introduction of root locus	137
	5.1.1 Angle criteria and magnitude criteria	138
5.2	Relationship between root loci and time response of system	140
	5.2.1 Location of roots of characteristics equation in S–plane	140
	5.2.2 Relationship	141
5.3	Rules for manual calculation and construction of root locus, analysis and design using root locus concept, stability analysis using R–H criteria	142
5.3.1	Effect of addition of poles and zeros on a root locus	151

5.1 INTRODUCTION OF ROOT LOCUS

Root locus is a graphical method in which roots of a characteristics equation are plotted in S–plane for the different values of parameter. The locus of the roots of a characteristics equation when gain is varied from zero to infinity is called root locus.

Consider a unity feedback system



where, K is the gain.

Now, characteristics equation is;

$$1 + G(S)\,H(S) = 0$$

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)\,H(S)}$$

and, $1 + G(S) \cdot H(S) = 0$

or, $1 + \frac{K}{S(S+2)} = 0$

or, $S(S + 2) + K = 0$

or, $S^2 + 2S + K = 0$

or, $S = \frac{-2 \pm \sqrt{(2)^2 - 4K}}{2}$

Scanned with CamScanner

or, $S = -1 \pm \sqrt{1-K}$

or, $S_1 = -1 + \sqrt{1-K}$

and, $S_2 = -1 - \sqrt{1-K}$

When we plot these roots by varying the value of K from 0 to ∞, We obtain root locus.

Case 1

When K = 0; then,

$S_1 = -1 + \sqrt{1-K} = -1 + 1 = 0$

and, $S_2 = -1 - \sqrt{1-K} = -1 - 1 = -2$

These roots are known as 'open loop poles'

Case 2

When $0 < K < 1$;

$S_1 = -1 + \text{Something}$

$S_2 = -1 - \text{Something}$

The roots are real, distinct and unequal.

Case 3

When K = 1;

$S_1 = S_2 = -1$

Both roots are at (−1).

The roots are real and equal.

Case 4

When K > 1;

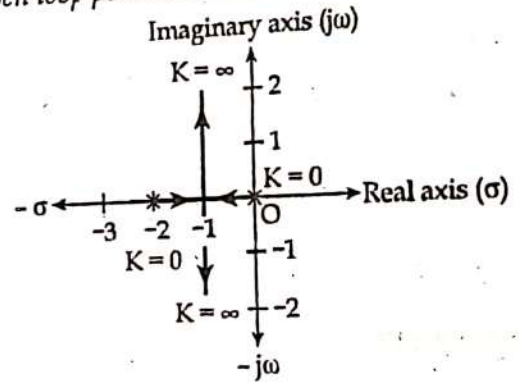
$S_1 = -1 + j \cdot \text{Something}$

$S_2 = -1 - j \cdot \text{Something}$

The roots are complex and unequal.

NOTE

The root locus always starts form open loop poles and ends with zeros or infinity.



In root locus;

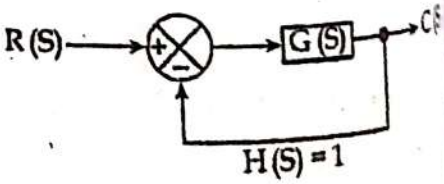
- Poles are denoted by sign × (cross)
- Zeros are denoted by zero 0 (zero)
- Multiple poles are denoted by sign * (asterisk)

5.1.1 Angle Criteria and Magnitude Criteria

For a system, characteristics equation is;

$1 + G(S)H(S) = 0$

or, $G(S)H(S) = -1$



Taking magnitude on both sides; we have,

$|G(S)H(S)| = |-1|$

$\therefore$ $\boxed{|G(S)H(S)| = 1}$; which is the **magnitude criteria.**

Taking angle on both sides; we have,

$\angle G(S)H(S) = \angle(-1)$

or, $\boxed{\angle G(S)H(S) = \pm 180°(2m+1)}$; which is known as **angle criteria.**

where, m = 0, 1, 2, 3, ...

- Value of S that satisfies both angle and magnitude criteria are the roots of characteristics equation or closed loop poles.

Scanned with CamScanner

- Locus of point in the complex plane satisfying only angle criteria only is called root locus.
- The value of gain (K) can be determined form magnitude criteria.

EXAMPLE 5.1

Consider a system with $G(S)H(S) = \frac{K}{S(S+4)}$. Find whether $S = -3$ is on root locus or not. Find the value of K for which $S = -3$ is a root of characteristics equation.

Solution:

Here,

$$G(S)H(S) = \frac{K}{S(S+4)}$$

Checking whether $S = -3$ lies on root locus or not;

$$\angle G(S)H(S) = \left.\frac{\angle K}{\angle S\,\angle(S+4)}\right|_{S=-3} = \frac{\angle K}{\angle(-3)\,\angle 1} = \frac{0^\circ}{(180^\circ)\,0^\circ} = \frac{0^\circ}{180^\circ + 0^\circ} = \frac{0^\circ}{180^\circ} = -180^\circ$$

Here, angle criteria is satisfied, so, $S = -3$ lies on root locus.

If $S = -3$ is a root of characteristics equation; then, it have to satisfy both magnitude and angle criteria.

Magnitude criteria gives;

$$|G(S)H(S)| = 1$$

or, $$\left|\frac{K}{S(S+4)}\right|_{S=-3} = 1$$

or, $$\frac{|K|}{|-3||-3+4|} = 1$$

or, $$\frac{|K|}{3 \times 1} = 1$$

$\therefore$ $K = 3$

EXAMPLE 5.2

For the unity feedback system with open loop transfer function $G(S) = \frac{K}{(S+1)(S+3)}$, Use angle criteria to check whether the root locus passes from point $S_d = -2 + j3.5$. If yes, use magnitude criteria to select the appropriate value of gain parameter. **[2071 Chaitra]**

Solution:

Here,

$$\text{OLTF; } G(S) = \frac{K}{(S+1)(S+3)}$$

If $S_d = -2 + j3.5$ satisfy angle criteria; then, passes from root locus.

$$\angle G(S)H(S) = \left.\frac{\angle K}{\angle(S+1)\angle(S+3)}\right|_{S_d=-2+j3.5}$$

or, $$\angle G(S)H(S) = \frac{0^\circ}{\angle(-2+1+j3.5)\angle(-2+3+j3.5)} = \frac{0^\circ}{\angle(-1+j3.5)\angle(1+j3.5)}$$

$$= \frac{0^\circ}{(105.95^\circ)(74.05^\circ)}$$

$$= \frac{0^\circ}{180^\circ} = -180^\circ; \text{ which satisfy angle criteria.}$$

Now, using magnitude criteria;

$$|\angle G(S)H(S)| = 1$$

Scanned with CamScanner

or, $\left.\frac{K}{(S+1)(S+3)}\right|_{S_d=-2+j3.5} = 1$

or, $\frac{|K|}{|(-1+j3.5)||(1+j3.5)|} = 1$

or, $\frac{K}{\left(\frac{\sqrt{53}}{2}\right)\left(\frac{\sqrt{53}}{2}\right)} = 1$

$\therefore \quad K = \frac{53}{4} = 13.25$

5.2 RELATIONSHIP BETWEEN ROOT LOCI AND TIME RESPONSE OF SYSTEM

5.2.1 Location of Roots of Characteristics Equation in S–Plane

For second degree system;

$$\text{Transfer function (T.F.)} = \frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

and, characteristics equation is;

$$S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$$

or, $(S + \zeta\omega_n)^2 - \zeta^2\omega_n^2 + \omega_n^2 = 0$

or, $(S + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2) = 0$

or, $(S + \zeta\omega_n)^2 = -\omega_n^2(1 - \zeta^2)$

or, $(S + \zeta\omega_n)^2 = \left(\pm j\omega_n\sqrt{1 - \zeta^2}\right)^2$

or, $S + \zeta\omega_n = \pm j\omega_n\sqrt{1 - \zeta^2}$

$\therefore \quad S = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$

Here, we plot all the roots in following cases:

i) Undamped case ($\zeta = 0$)

$S_1 = (0, \omega_n)$
$S_2 = (0, -\omega_n)$
; indicates all roots are in imaginary ($j\omega_n$) axis *i.e.*, in y–axis.

ii) Critically damped case ($\zeta = 1$)

$S_1 = (-\omega_n, 0)$
$S_2 = (-\omega_n, 0)$
Roots are in negative real axis *i.e.*, in (– x–axis)

iii) Over-damped case ($\zeta > 1$)

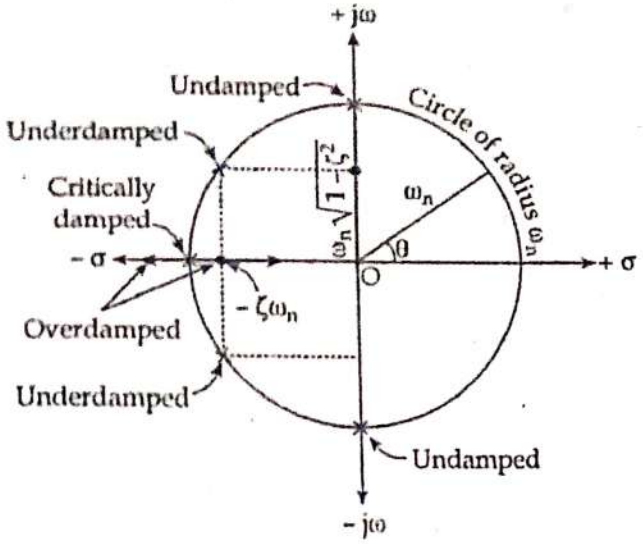
$S_1 = -\zeta\omega_n + j\omega_n \times$ Imaginary term
$= -\zeta\omega_n -$ Constant
$S_2 = -\zeta\omega_n - j\omega_n \times$ Imaginary term
$= -\zeta\omega_n +$ Constant
; Some points around in real axis *i.e.*, x–axis.

iv) Under-damped case ($0 < \zeta < 1$)

$S_1 = \left(-\zeta\omega_n, \omega_n\sqrt{1 - \zeta^2}\right)$
$S_2 = \left(-\zeta\omega_n, -\omega_n\sqrt{1 - \zeta^2}\right)$
; in X–Y plane.

Now, drawing these roots on graph; we get,

Scanned with CamScanner



$S = \pm\sigma \pm j\omega$

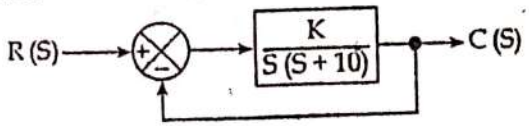
Real axis

Imaginary axis

5.2.2 Relationship

Form root locus we can define the time response of the given system.

Consider a system characteristics equation;



$$1 + G(S)H(S) = 0$$

or, $$1 + \frac{K}{S(S+10)} = 0$$

or, $$S(S+10) + K = 0$$

or, $$S^2 + 10S + K = 0$$

or, $$S = \frac{-10 \pm \sqrt{(10)^2 - 4K}}{2} = -5 \pm \sqrt{25 - K}$$

$\therefore$ $$S_1 = -5 - \sqrt{25 - K}$$

and, $$S_2 = -5 + \sqrt{25 - K}$$

Now, varying value of K from zero to higher; we have,

K	Pole 1 (S_1)	Pole 2 (S_2)	Comment
0	-10	0	Over damped
5	-9.47	-0.53	
10	-8.87	-1.13	
15	-8.16	-1.84	
20	-7.24	-2.76	
25	-5	-5	Critically damped
30	$-5 + j2.23$	$-5 - j2.23$	Under damped
35	$-5 + j3.16$	$-5 - j3.16$	
40	$-5 + j3.87$	$-5 - j3.87$	
45	$-5 + j4.47$	$-5 - j4.47$	
50	$-5 + j5$	$-5 - j5$	

Here,

For lower value of K $\longrightarrow$ Over damped ($\zeta > 1$)

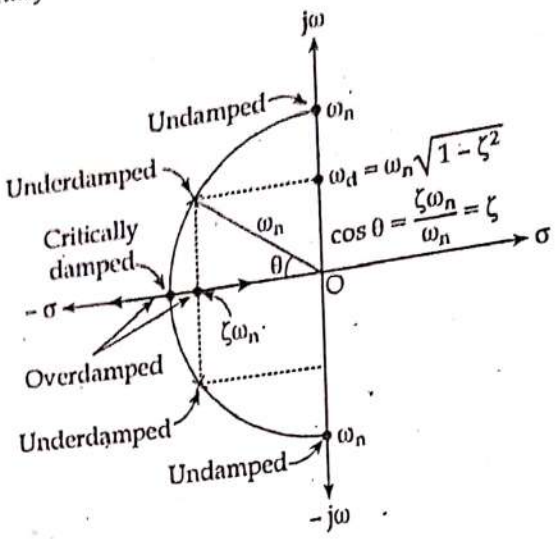
For medium value of K $\longrightarrow$ Critically damped ($\zeta = 1$)

Scanned with CamScanner

For higher value of K ⟶ Under damped $(0 < \zeta < 1)$

If root locus crosses only imaginary axis for any value of K ⟶ undamped case $(\zeta = 0)$.

Analysis



We know that;

Peak time $(t_p) = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \dfrac{\pi}{\omega_d}$

Maximum percentage overshoot $(M_p) = e^{\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)} \times 100\%$

Damped frequency $(\omega_d) = \omega_n\sqrt{1-\zeta^2}$

Settling time $(T_s) = \dfrac{4}{\zeta\omega_n}$

When K increase (Under-damped), then angle θ increases, then cos θ decreases, means damping ratio (ζ) decreases, which means maximum overshoot increases.

Decreases in ζ means damped frequency (ω_d) increases so that peak time (t_p) decreases. Settling time (T_s) depends upon negative real part $(-\zeta\omega_n)$.

i.e., $K\uparrow, \theta\uparrow, \cos\theta\downarrow, \zeta\downarrow, M_p\uparrow, \omega_d\uparrow, t_p\downarrow$

5.3 RULES FOR MANUAL CALCULATION AND CONSTRUCTION OF ROOT LOCUS, ANALYSIS AND DESIGN USING ROOT LOCUS CONCEPT, STABILITY ANALYSIS USING R–H CRITERIA

Rules for drawing root locus

Rule 1

The root locus is symmetrical about real axis.

Rule 2

The root locus starts form open loop pole (at K = 0) and terminate to open loop zero (at K = ∞).

EXAMPLE 5.3

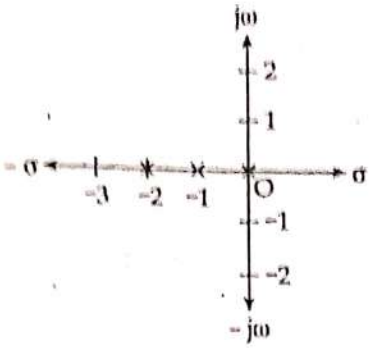
Consider the system with open loop transfer function $G(S) = \dfrac{1}{S(S+1)(S+2)}$, $H(S) = 1$. Sketch Root locus of given system.

Rule 3 Determine open loop poles and zeros and plot them on graph

- The pole is indicated by sign ×.
- The zero is indicated by sign 0.

Scanned with CamScanner

For above question;
Open loop poles at $S_1 = 0$, $S_2 = -1$ and $S_3 = -2$.
No open loop zeros.
Now plotting them;



Rule 4

The number of branch of root locus is equal to number of open loop poles or zeros whichever is greater.

If N is the number of separate root loci (branches).
P is the number of finite open loop poles.
Z is the number of finite open loop zeros.

Then,

$N = P$ if $P > Z$
$N = Z$ if $Z > P$
$N = P = Z$ if $P = Z$

From the above question;

Number of open loop poles (P) = 3
Number of open loop zeros (Z) = 0
∴ Number of root loci (N) = P = 3 since, P > Z.

Rule 5

Any point on real axis is a part of a Root locus if and only if number of poles and zeros to its right is odd.

In above question;

Staying at 0 : Number of poles and zeros to its right are zero.
Staying at −1 : Number of poles and zeros to its right is one which is odd. So, root locus lies in between 0 to −1.
Staying at −2 : Number of poles and zeros to its right is two, which is even; so, no root locus lie.

- Therefore, root locus lies in between (0 to −1)
- Staying at away from −2, the number of poles and zeros to its right is three, which is odd.

Therefore, root locus lies in between −2 to −∞.

Rule 6 Centroid of asymptote

Asymptotes are straight line along which root locus tends to infinity. In other words, the branches of root locus tend to infinity along set of straight line is known as asymptotes. The point of intersection of asymptotes with real axis is called centroid of asymptotes (σ_A) and is given by;

$$\sigma_A = \frac{\text{Sum of poles} - \text{Sum of zeros}}{P - Z}$$

In the above question;

Sum of poles = [0 + (−1) + (−2)] = −3
Sum of zeros = 0

Then,

Scanned with CamScanner

$$\sigma_A = \frac{-3-0}{3-0} = \frac{-3}{3} = -1$$

i.e., centroid at S = −1.

Rule 7 Number of asymptotes and angle of asymptotes

Total number of asymptotes = P − Z

and, Angle of asymptote $(\phi) = \frac{(2m+1)\times 180^\circ}{P-Z}$

This is the angle made with positive real axis.

and, m = 0, 1, 2, ...

For one asymptote; put m = 0 only

For two asymptote; put m = 0 and 1 only

For three asymptote; put m = 0, 1 and 2 only and so on.

For above question;

Total number of asymptotes = P − Z = 3 − 0 = 3

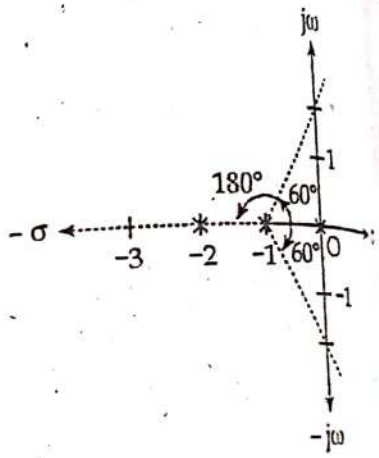
Angle of asymptotes $(\phi) = \frac{(2m+1)\times 180^\circ}{P-Z}$

Put, m = 0, 1 and 2; then,

$$\phi_1 = \frac{(2\times 0+1)\times 180^\circ}{3} = 60^\circ \text{ at } m = 0$$

$$\phi_2 = \frac{(2\times 1+1)\times 180^\circ}{3} = 180^\circ \text{ at } m = 1$$

$$\phi_3 = \frac{(2\times 2+1)\times 180^\circ}{3} = 300^\circ \text{ at } m = 2$$



Rule 8 Angle of departure (only for complex pole) and angle of arrival/aperture (only for complex zero) of the root loci

Angle of departure (ϕ_d) = 180° − Sum of angles of vectors drawn to these poles from other poles + Sum of angles of vectors drawn to this pole from zeros

Angle of aperture (ϕ_a) = 180° − Sum of angles of vectors drawn to this zero from other zeros + Sum of angles of vectors drawn to this zero from poles

NOTE

i) Angle of departure (ϕ_d) is only for complex pole

ii) Angle of aperture (ϕ_a) is only for complex zero. /angle of arrival

iii) All the angles are measured with positive real axis.

Rule 9 Rule axis 'break away' or 'break in point'

i) If root locus lies between two open loop poles then there is at least one break away point. Breakaway point is that point where root locus leaves real axis. [Figure (i)]

ii) If root locus lies is between two open loop zeros then at least one break in point. [Figure (ii)]

iii) If root locus lies in between open loop poles and open loop zeros then either there will be no break in and breakaway point or both point occur. [Figure (iii)]

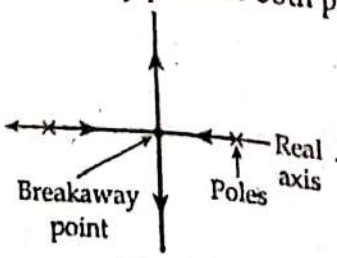
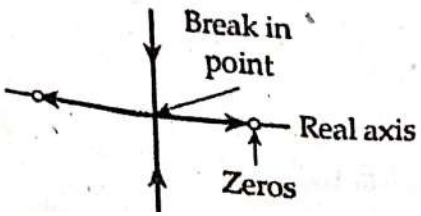


Figure: (i)



Scanned with CamScanner

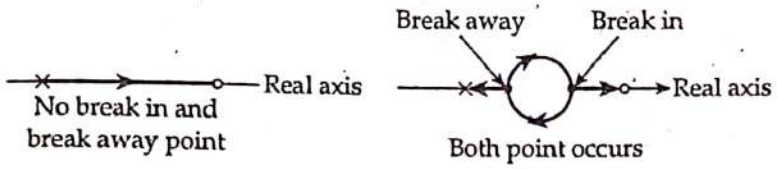


Figure: (iii)

iv) The breakaway or break in point can be determined by $\frac{dK}{dS} = 0$ in characteristics equation.

In above question;

$$G(S)H(S) = \frac{K}{S(S+1)(S+2)}$$

and, characteristics equation is;

$$1 + G(S)H(S) = 0$$

or, $1 + \frac{K}{S(S+1)(S+2)} = 0$

or, $S(S+1)(S+2) + K = 0$

or, $K = -S(S+1)(S+2) = -(S^3 + 3S^2 + 2S)$

Taking derivative and $\frac{dK}{dS} = 0$; we have,

so, $\frac{dK}{dS} = -(3S^2 + 6S + 2) = 0$

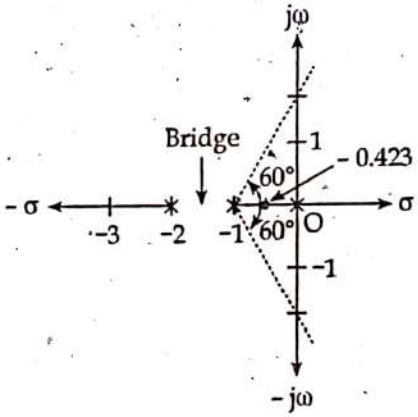
or, $3S^2 + 6S + 2 = 0$

On solving; we get,

or, $S = -0.423, -1.577$

↓ ↓

Valid Invalid



The root locus lies between (0 to −1) and (−2 to −∞) So, (−1.577) root lies on the bridge and hence invalid.

Plotting these points on graph as shown aside;

Since, the point S = −0.423 lies in between two poles so that point is break away point.

Rule 10 Use R–H criteria to determine point of intersection, of root loci with imaginary (Jω) axis

For above question; characteristics equation is;

$$S(S+1)(S+2) + K = 0$$

or, $S^3 + 3S^2 + 2S + K = 0$

Forming R-H array;

S^3	1	2
S^2	3	K
S^1	b_1	
S^0	C_1	

$$b_1 = -\frac{1}{3}\begin{vmatrix} 1 & 2 \\ 3 & K \end{vmatrix} = -\left(\frac{K-6}{3}\right) = \frac{6-K}{3}$$

C_1 = Constant term = $K \neq 0$

so, R-H array becomes;

S^3	1	2
S^2	3	K
S^1	$\frac{6-K}{3}$	
S^0	K	

Scanned with CamScanner

For jω crossing (marginal stability); any row should be zero.

so, $\frac{6-K}{3} = 0$

i.e., $K = 6$

so, the auxiliary equation is;

$A(S) = 3S^2 + K$

Substituting $A(S) = 0$; we have,

$3S^2 + 6 = 0$

or, $S^2 = -2$

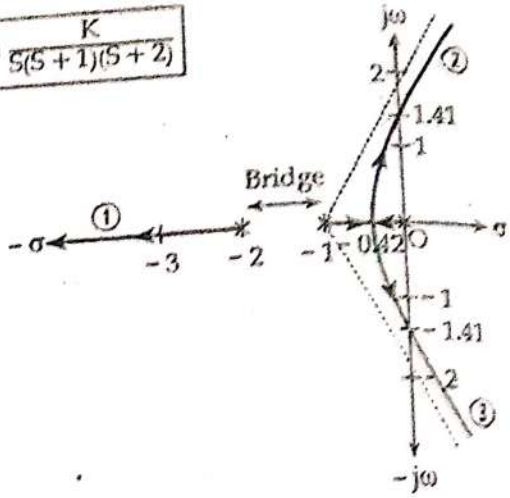
or, $S^2 = (\pm j\sqrt{2})^2$

∴ $S = \pm j\sqrt{2}$

i.e., $S = \pm j1.41$ is the crossing.

Now, the graph becomes as shown aside.

$G(S) = \frac{K}{S(S+1)(S+2)}$



EXAMPLE 5.4

For unity feedback system, the open loop transfer function is $G(S) = \frac{K}{S(S+2)(S^2+6S+25)}$

i) Sketch root locus.

ii) At what value of K, system becomes stable.

iii) At this point, determine the frequency of oscillation.

iv) Find dominant pole when $\zeta = 0.5$ [2066 Magh, 2073 Chaitra]

Solution:

Step 1 Find open loop poles and plot it on graph

Here,

$$G(S) = \frac{K}{S(S+2)(S^2+6S+25)}$$

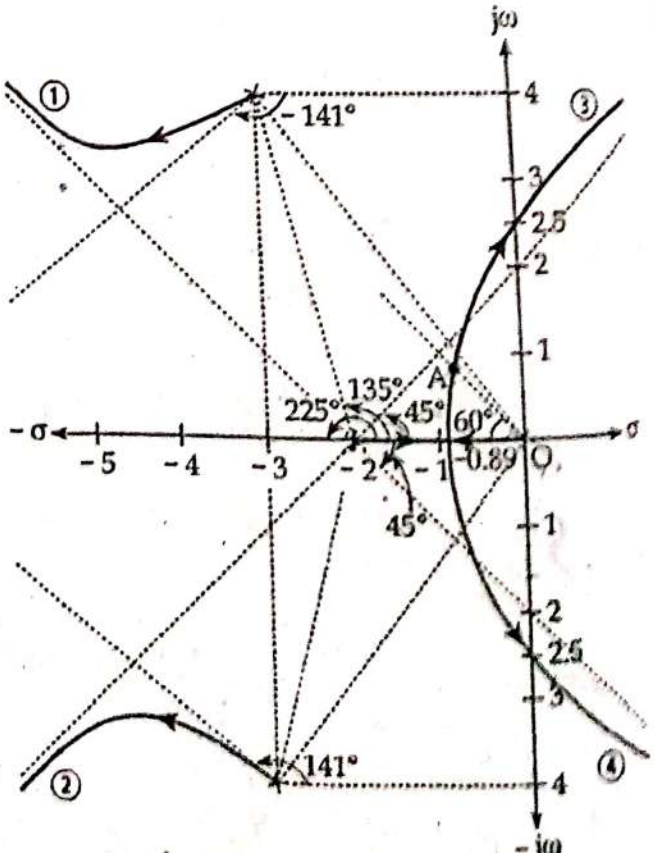
$G(S) = \frac{K}{S(S+2)(S^2+6S+25)}$

Poles at; $S_1 = 0$

$S_2 = -2$

$S_3 = -3 + j4$

$S_4 = -3 - j4$



Step 2

Let, P is the number of open poles = 4

Z is the number of open loop zeros = 0

∴ Number of root loci = 4; since, P > Z

Step 3

Root locus lies between (0 to −2)

Step 4 Centroid of asymptote

$$\sigma_A = \frac{\text{Sum of poles} - \text{Sum of zeros}}{P - Z}$$

$$= \frac{0 - 2 - 3 + j4 - 3 - j4}{4 - 0}$$

$$= -2$$

i.e., centroid at $\sigma_A = -2$.

Scanned with CamScanner

Step 5

Here,

Number of asymptotes = P − Z = 4 − 0 = 4

Angle of asymptote $(\phi) = \frac{(2m+1)\times 180°}{P-Z}$

where, m = 0, 1, 2, 3, ...

Now,

$$\phi_1 = \frac{(2\times 0+1)\times 180°}{4} = 45° \text{ at } m = 0$$

$$\phi_2 = \frac{(2\times 1+1)\times 180°}{4} = 135° \text{ at } m = 1$$

$$\phi_3 = \frac{(2\times 2+1)\times 180°}{4} = 225° \text{ at } m = 2$$

$$\phi_4 = \frac{(2\times 3+1)\times 180°}{4} = 315° \text{ at } m = 3$$

Plotting these angles from centroid.

Step 6 Angle of departure (ϕ_d) for upper complex pole

Angle mode by S = 0 pole at upper complex pole;

$$\theta_1 = 90° + \tan^{-1}\left(\frac{3}{4}\right) = 126.87°$$ because in 2nd quadrant (90+α)

From S = −2 pole at upper complex pole;

$$\theta_2 = 90° + \tan^{-1}\left(\frac{1}{4}\right) = 104.036°$$

Angle mode by lower complex pole to upper;

$$\theta_3 = 90°$$

From formula; we have,

$$\phi_d = 180° - (126.87° + 104.036° + 90°) = -140.90° \approx -141°$$

Plot a line from upper complex pole. The root locus has a direction through this angle. Similarly, angle of departure for lower complex pole $\phi_d = +141°$ since root locus are symmetrical to real axis.

Plot a line from lower complex pole.

Step 7 Breakaway point

We have characteristics equation;

$$1 + G(S)H(S) = 0$$

or, $$1 + \frac{K}{S(S+2)(S^2+6S+25)} = 0$$

or, $$S(S+2)(S^2+6S+25) + K = 0$$

or, $$S^4 + 8S^3 + 37S^2 + 50S + K = 0$$

or, $$K = -(S^4 + 8S^3 + 37S^2 + 50S)$$

Perform $\frac{dK}{dS} = 0$; we have,

or, $$-(4S^3 + 24S^2 + 74S + 50) = 0$$

or, $$2S^3 + 12S^2 + 37S + 25 = 0$$

Using calculator; we have,

$$S_1 = -0.8981 \text{ (Valid)}$$

Scanned with CamScanner

$S_{2,3} = -2.55 \pm j2.722$ (Invalid)

Since, the root locus lies between (0 to −2). The point S = −0.89 is valid and known as breakaway point (because it lies between two open loop poles.)

Step 8 jω crossing

Whether the root locus crosses imaginary axis or not we use R-H criteria.

The characteristics equation is;

$$1 + G(S)H(S) = 0$$

or, $S^4 + 8S^3 + 37S^2 + 50S + K = 0$

Now, R-H array becomes;

S^4	1	37	K
S^3	8	50	0
S^2	b_1	b_2	

$$b_1 = -\frac{1}{8}\begin{vmatrix} 1 & 37 \\ 8 & 50 \end{vmatrix} = -\left(\frac{50 - 8 \times 37}{8}\right) = 30.75$$

$$b_2 = -\frac{1}{8}\begin{vmatrix} 1 & K \\ 8 & 0 \end{vmatrix} = K\left(\frac{8}{8}\right) = K$$

Again,

S^4	1	37	K
S^3	8	50	0
S^2	30.75	K	
S^1	C_1		
S^0	d_1		

$$C_1 = -\frac{1}{30.75}\begin{vmatrix} 8 & 50 \\ 30.75 & K \end{vmatrix} = \left(\frac{1537.5 - 8K}{30.75}\right)$$

and, $d_1 = K \neq 0$

So, complete R-H table becomes;

S^4	1	37	K
S^3	8	50	0
S^2	30.75	K	
S^1	$\left(\frac{1537.5 - 8K}{30.75}\right)$		
S^0	K		

For jω-crossing; one row must be zero which is only possible for S^1 row.

i.e., $\frac{1537.5 - 8K}{30.75} = 0$

Solving; we get,

$K = 192.18$

and, the auxiliary equation is;

$$A(S) = 30.75S^2 + K$$

Substituting; we have,

$$30.75S^2 + K = 0$$

or, $30.75S^2 + 192.18 = 0$

$\therefore$ $S = \pm j2.5$

Therefore, the root locus crosses the jω axis at ±2.5.

Answer:

ii) At K = 192.18; the system is marginally stable. For stable the range of K is (0 < K < 192.18).

iii) Frequency of oscillation (ω) = 2.5 rad/sec.

iv) Dominant poles when ζ = 0.5

We know that;

$\cos\theta = \zeta$

or, $\cos\theta = 0.5$

$\therefore$ $\theta = 60°$

The dominant pole can be found out by plotting this angle in second quadrant.

The root locus meets this line at point A.

By measuring scale the dominant pole is A(−0.75 + j0.76).

We can find out value of K at this point by using magnitude criteria, or by graphical method.

EXAMPLE 5.5

Sketch the root locus of system with open loop transfer function $G(S) = \frac{K(S+2)}{S^2 + 2S + 3}$ and H(S) = 1; where K ≥ 0.

Solution:

Step 1

Open loop poles are found by;

$$S^2 + 2S + 3 = 0$$

$\therefore$ $S_1 = -1 + j\sqrt{2}$

and, $S_2 = -1 - j\sqrt{2}$

Open loop zeros is at $S_3 = -2$ and plot these on graph.

Step 2

Number of open loop poles (P) = 2

Number of open loop zeros (Z) = 1

Since, P > Z; then,

Number of root locus = P

$\therefore$ N = 2

Step 3

Root locus lies in between −2 to −∞

Step 4

$$\text{Centroid of asymptotes } (\sigma_A) = \frac{\sum P - \sum Z}{P - Z} = \frac{(-1 + j\sqrt{2} - 1 - j\sqrt{2}) - (-2)}{2 - 1} = 0$$

Step 5

Number of asymptotes = P − Z = 2 − 1 = 1

and, $\text{Angle of asymptote } (\phi) = \frac{(2m+1) \times 180°}{P - Z} = \frac{(2 \times 0 + 1) \times 180°}{1} = 180° \text{ at } m = 0$

Scanned with CamScanner

Step 6 Angle of departure (ϕ_d) for upper complex pole;

Angle made by lower complex pole to upper complex pole $(\theta_1) = +90°$

Angle mode by zeros to upper complex pole $(\theta_2) = \tan^{-1}\left(\frac{\sqrt{2}}{1}\right) = 54.73°$

$\therefore \quad \phi_d = 180° - 90° + 54.73° = 144.73° \approx 145°$

Now, plotting this angle form upper complex pole;

The angle of departure (ϕ_d) for lower complex pole is just mirror of upper.

$\therefore \quad \phi_d = -144.73° \approx -145°$

Step 7 Calculation of break in point

The characteristics equation is;

$$1 + G(S)H(S) = 0$$

or, $$1 + \frac{K(S+2)}{S^2+2S+3} = 0$$

or, $$S^2 + 2S + 3 + K(S+2) = 0$$

or, $$K = -\frac{S^2+2S+3}{S+2}$$

Now taking $\frac{dK}{dS} = 0$; we have,

$$\frac{dK}{dS} = -\left[\frac{(S+2)(2S+2) - (S^2+2S+3)}{(S+2)^2}\right] = 0$$

or, $$(S+2)(2S+2) - (S^2+2S+3) = 0$$

or, $$2S^2 + 6S + 4 - (S^2+2S+3) = 0$$

or, $$S^2 + 4S + 1 = 0$$

Solving on calculator; we have,

$S_1 = -0.268$ (Invalid)

$S_2 = -3.73$ (Valid)

Since, the root locus lies between -2 to $-\infty$, the break in point lies at $S = -3.73$.

Step 8 jω crossing

The characteristics equation is;

$$S^2 + 2S + 3 + KS + 2K = 0$$

or, $$S^2 + (2+K)S + 3(3+2K) = 0$$

The R-H array becomes;

S^2	1	$(3+2K)$
S^1	$(2+K)$	
S^0	$(3+2K)$	

The S^0 row can't be zero. The only possible row to be zero is S^1 row.

i.e., $2 + K = 0$

and, $K = -2$

Now, the auxiliary equation is;

$$A(S) = S^2 + (3+2K) = 0$$

Put, $K = -2$; then,

$$S^2 + [3 + 2 \times (-2)] = 0$$

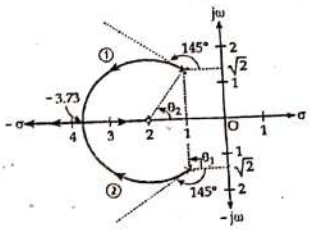
or, $$S^2 - 1 = 0$$

or, $$S^2 = 1$$

or, $$S^2 = (\pm 1)^2$$

$\therefore \quad S = \pm 1$

Here, $S = \pm 1$ and it is a real part so no jω crossing.



NOTE

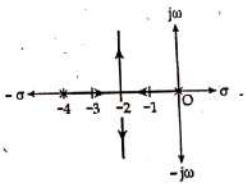
Generally, a root locus is circle with single asymptote.

5.3.1 Effect of Addition of Poles and Zeros on a Root Locus

I) Effect of addition of poles

Let, $G(S)H(S) = \frac{K}{S(S+4)}$

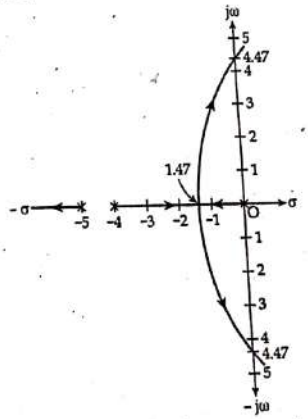
The root locus for this system is;



Now add one pole $(S = -5)$; then,

$$G'(S)H'(S) = \frac{K}{S(S+4)(S+5)}$$

The root locus for this system will be;



After addition of poles, the two branches of root locus moves to the right, and system stability decreases. For further addition of poles to the left half, the breaking point moves towards right. Hence,

Scanned with CamScanner

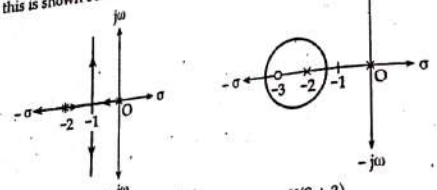
we can concluded that addition of poles to left half, the root locus shifted towards right hand side and stability of the system decreases.

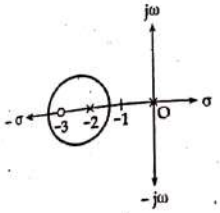
ii) Addition of zeros

Now, consider open loop transfer function;

$$G(S)H(S) = \frac{K}{S(S+2)}$$

The root locus for this is shown below:





After addition of zeros to left half at S = −3; the OLTF is $G(S) = \frac{K(S+3)}{S(S+2)}$.

It can be seen that by addition of zero towards left, the root locus shifted towards left half. Since, root locus shifts towards left half, the relative stability increases.

5.4 SOLVED PROBLEMS

1. A unity feedback system has an open loop transfer function $G(S) = \frac{K(S+1)}{S(S-1)}$. Sketch the root locus plot with 'K' as variable parameter and show that the locl of complex roots are part of circle with (−1, 0) as center and radius = $\sqrt{2}$.

Solution:

Step 1

Here,

Open loop poles at $S_1 = 0, S_2 = 1$

Open loop zeros at $S_3 = -1$

Now, plotting this on graph.

Step 2

Number of open loop poles (P) = 2

Number of open loop zeros (Z) = 1

Since, P > Z

so, Number of root locus (N) = P = 2

Step 3

Root locus lies in between 0 to 1 and −1 to −∞.

Step 4

$$\text{Centroid of asymptotes } (\sigma_A) = \frac{\sum P - \sum Z}{P - Z} = \frac{(0+1)-(-1)}{2-1} = 2$$

Step 5

Number of asymptotes = P − Z = 2 − 1 = 1

$$\text{Angle of asymptote } (\phi) = \frac{(2m+1)\times 180^\circ}{P-Z}$$

and, $\phi_1 = \frac{(2\times 0+1)\times 180^\circ}{1} = 180^\circ$

Step 6

Here, no complex pole and zeros so no need to calculate angle of departure and aperture.

Step 7 Calculation of break in and break away point

The characteristics equation is;

$$1 + G(S)H(S) = 0$$

or, $1 + \frac{K(S+1)}{S(S-1)} = 0$

or, $S(S-1) + K(S+1) = 0$

or, $K = -\frac{S^2 - S}{S+1}$

Now, taking derivative; we have,

$$\frac{dK}{dS} = 0$$

or, $\frac{dK}{dS} = -\frac{(S+1)(2S-1) - (S^2 - S)}{S+1} = 0$

Scanned with CamScanner

or, $(S+1)(2S-1)-(S^2-S)=0$

or, $2S^2+S-1-S^2+S=0$

or, $S^2+2S-1=0$

Now, solving through calculator; we have,

$S_1 = 0.43$

and, $S_2 = -2.43$

Since, root locus lies between (0 to 1) and (−1 to −∞) both are valid.

Step 8 Jω crossing

The characteristics equation is;

$1+G(S)H(S)=0$

or, $S^2-S+K(S+1)=0$

or, $S^2-S+KS+K=0$

or, $S^2-S(K-1)+K=0$

Now, R-H array becomes;

S^2	1	K
S^1	(K − 1)	0
S^0	K	

The S^0 rows can't be zero.

The only possible row to be zero is S^1 row.

so, $K-1=0$

or, $K=1$

The auxiliary equation is;

$A(S)=(S^2+K)=0$

Substituting; we have,

$S^2+K=0$

or, $S^2+1=0$

or, $S^2=-1$

or, $S^2=(\pm j)^2$

∴ $S=\pm j$

∴ Root locus crosses imaginary axis in ± j.

Mathematical Method

$$G(S)H(S)=\frac{K(S+1)}{S(S-1)}$$

Put, $S=\sigma+j\omega$;

so, $$G(S)H(S)=\frac{K(\sigma+j\omega+1)}{(\sigma+j\omega)(\sigma+j\omega-1)}$$

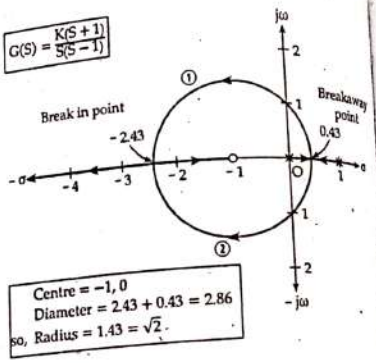
or, $$G(j\omega)H(j\omega)=\frac{K(\sigma+1+j\omega)}{(\sigma+j\omega)(\sigma-1+j\omega)}$$

Now,

$$\angle G(j\omega)\angle H(j\omega)=\frac{\angle K\angle(\sigma+1+j\omega)}{\angle(\sigma+j\omega)\angle(\sigma-1+j\omega)}=\tan^{-1}\left(\frac{\omega}{\sigma+1}\right)-\tan^{-1}\left(\frac{\omega}{\sigma}\right)-\tan^{-1}\left(\frac{\omega}{\sigma-1}\right)$$

To prove root locus, it must satisfy angle criteria;

$G(S)=\frac{K(S+1)}{S(S-1)}$



Centre = −1, 0
Diameter = 2.43 + 0.43 = 2.86
so, Radius = 1.43 = √2.

so, $$\tan^{-1}\left(\frac{\omega}{\sigma+1}\right)-\tan^{-1}\left(\frac{\omega}{\sigma-1}\right)-\tan^{-1}\left(\frac{\omega}{\sigma}\right)=180°$$

or, $$\tan^{-1}\left(\frac{\omega}{\sigma+1}\right)-\tan^{-1}\left(\frac{\omega}{\sigma-1}\right)=180°+\tan^{-1}\left(\frac{\omega}{\sigma}\right)$$

Taking tan on both sides; we have,

$$\tan\left[\tan^{-1}\left(\frac{\omega}{\sigma+1}\right)-\tan^{-1}\left(\frac{\omega}{\sigma-1}\right)\right]=\tan\left[180°+\tan^{-1}\left(\frac{\omega}{\sigma}\right)\right]$$

or, $$\frac{\left(\frac{\omega}{\sigma+1}\right)-\left(\frac{\omega}{\sigma-1}\right)}{1+\left(\frac{\omega}{\sigma+1}\right)\left(\frac{\omega}{\sigma-1}\right)}=\frac{\omega}{\sigma}$$

or, $$\frac{\omega(\sigma-1)-\omega(\sigma+1)}{(\sigma^2-1)+\omega^2}=\frac{\omega}{\sigma}$$

or, $$\frac{\omega\sigma-\omega-\omega\sigma-\omega}{(\sigma^2-1)+\omega^2}=\frac{\omega}{\sigma}$$

or, $$\frac{-2\omega}{(\sigma^2-1)+\omega^2}=\frac{\omega}{\sigma}$$

or, $$\frac{-2}{(\sigma^2-1)+\omega^2}=\frac{1}{\sigma}$$

or, $-2\sigma=\sigma^2+\omega^2-1$

or, $\sigma^2+2\sigma+\omega^2-1=0$

or, $(\sigma^2+2\sigma+1)+\omega^2-2=0$

or, $(\sigma+1)^2+\omega^2=2$

or, $[\sigma-(-1)]^2+(\omega-0)^2=(\sqrt{2})^2$ (1)

Comparing equation (1) with standard equation of circle;

Centre = [−1, 0]

and, Radius = √2 unit

2. **Sketch the root locus for system having** $G(S)=\frac{K(S+1)}{(S^2+2S+2)(S^2+2S+5)}$ [2068 Chaitra]

Solution:

Step 1

Here,

$$G(S)=\frac{K(S+1)}{(S^2+2S+2)(S^2+2S+5)}$$

Now,

Open loop poles are at $S_{1,2}=-1\pm j$

Open loop zeros at $S_{3,4}=-1\pm 2j$

Plotting these poles and zeros on graph; we have,

Step 2

The root locus lies in between (−1 to −∞).

Step 3

Number of open loop poles (P) = 4

Number of open loop zeros (Z) = 1

since, P > Z; then,

Scanned with CamScanner

Number of root locus (N) = P = 4

Step 4 Centroid of asymptote

$$\sigma_A = \frac{\sum P - \sum Z}{P - Z} = \frac{1 + (-1+j) + (-1-j) + (-1+2j) + (-1-2j)}{4-1}$$

$$= -\left(\frac{3}{3}\right) = -1$$

$$G(S) = \frac{K(S+1)}{(S^2+2S+2)(S^2+2S+5)}$$

∴ Centroid (σ_A) at S = −1

Step 5

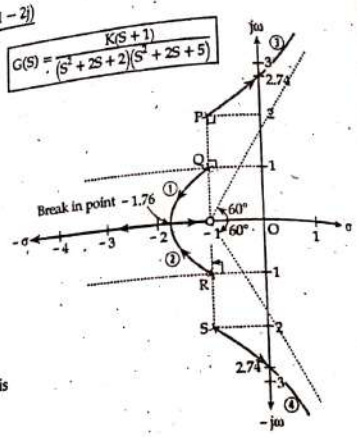
Number of asymptotes = P − Z = 4 − 1 = 3

Angle of asymptotes (ϕ) = $\frac{(2m+1)\times 180°}{P-Z}$

so, $\phi_1 = \frac{(2\times 0+1)\times 180°}{3} = 60°$ at m = 0

$\phi_2 = \frac{(2\times 1+1)\times 180°}{3} = 180°$ at m = 1

$\phi_3 = \frac{(2\times 2+1)\times 180°}{3} = 300°$ at m = 2



Step 6

Angle of departure (ϕ_d) for upper complex pole P is given by;

$\phi_{dP} = 180° - (90° + 90° + 90°) + 90° = 0°$

$= 0°(+)$

Angle of departure (ϕ_d) for lower complex S is just a mirror of ϕ_{dP} is given by;

$\phi_{dS} = 0°(-)$

Angle of departure for complex pole Q is given by;

$$\phi_{dQ} = 180° - \left(\underset{\text{From R}}{90°} + \underset{\text{From S}}{90°} + \underset{\text{From P}}{(-90°)}\right) + \underset{\text{From zero}}{90°} = 180°(+)$$

Angle of departure for complex pole R is just mirror of ϕ_{dQ} is given by;

$\phi_{dR} = 180°(-)$

Step 7 Break in point

The characteristics equation is;

$1 + G(S)H(S) = 0$

or, $1 + \frac{K(S+1)}{(S^2+2S+2)(S^2+2S+5)} = 0$

or, $(S^2+2S+2)(S^2+2S+5) + K(S+1) = 0$

or, $K = \frac{-(S^2+2S+2)(S^2+2S+5)}{(S+1)}$

Taking derivative; we have,

$\frac{dK}{dS} = 0$

Solving; we get,

$3S^4 + 12S^3 + 23S^2 + 22S + 4 = 0$

Using calculator; we have,

S = −0.23

or, $3S^3(S+0.23) + 11.31S^3 + 23S^2 + 22S + 4 = 0$

or, $3S^3(S+0.23) + 11.31S^2(S+0.23) + 20.3987S^2 + 22S + 4 = 0$

or, $3S^3(S+0.23) + 11.31S^2(S+0.23) + 20.3987S(S+0.23) + 17.3083(S+0.23) = 0$

or, $(S+0.23)(3S^3 + 11.31S^2 + 20.3987S + 17.3083) = 0$

Either, $S_1 = -0.23$ (Impossible)

Or, solving $3S^3 + 11.31S^2 + 20.3987S + 17.3083 = 0$ using calculator; we have,

$S_2 = -1.77$

$S_3 = -0.99 - j1.5$ } Impossible

$S_4 = -0.99 + j1.5$

Root locus lies between −1 to −∞; so, only S = −1.77 is valid and called as break in point.

Step 8 jω crossing can be calculated through R–H criteria

The characteristics equator is;

$S^4 + 4S^3 + 11S^2 + (14+K)S + (10+K) = 0$

S^4	1	11	(10 + K)
S^3	4	(14 + K)	0
S^2	b_1	b_2	

$$b_1 = -\frac{1}{4}\begin{vmatrix} 1 & 11 \\ 4 & 14+K \end{vmatrix} = \frac{30-K}{4}$$

$$b_2 = -\frac{1}{4}\begin{vmatrix} 1 & 10+K \\ 4 & 0 \end{vmatrix} = 10+K$$

The R-H array becomes;

S^4	1	11	(10 + K)
S^3	4	(14 + K)	0
S^2	$\left(\frac{30-K}{4}\right)$	(10 + K)	
S^1	C_1	C_2	
S^0	d_1		

$$C_1 = -\frac{1}{\left(\frac{30-K}{4}\right)}\begin{vmatrix} 4 & (14+K) \\ \left(\frac{30-K}{4}\right) & (10+K) \end{vmatrix} = \frac{260-K^2}{30-K}$$

$C_2 = 0$

and, d_1 = Constant = 10 + K

The final R-H array becomes;

S^4	1	11	(10 + K)
S^3	4	(14 + K)	0
S^2	$\left(\frac{30-K}{4}\right)$	(10 + K)	
S^1	$\left(\frac{260-K^2}{30-K}\right)$	0	
S^0	(10 + K)		

Scanned with CamScanner

Now, the only possible row for marginal stability is;

$$\frac{260 - K^2}{30 - K} = 0$$

or, $K^2 = 260$

or, $K^2 = (\pm 16.1245)^2$

$\therefore$ $K = 16.1245$ (only take positive value)

Since, root locus lies from $K = 0$ to $K = +\infty$.

and, the auxiliary equation is:

$$A(S) = \left(\frac{30 - K}{4}\right)S^2 + (10 + 16.1245) = 0$$

or, $(30 - K)S^2 + 104.498 = 0$

or, $(30 - 16.1245)S^2 = -104.498$

or, $S^2 = -\frac{104.498}{13.8755}$

or, $S^2 = (\pm j2.744)^2$

$\therefore$ $S = \pm j2.74$

The root locus crosses the imaginary axis at $= \pm j2.74$.

3. For unity feedback system with OLTF $G(S) = \frac{K(S+3)}{(S-1)(S^2+2S+4)}$, plot the root locus and find the range of gain parameter (K) for the system to be unstable, under-damped, and over damped. [2070 Ashadh]

Solution:

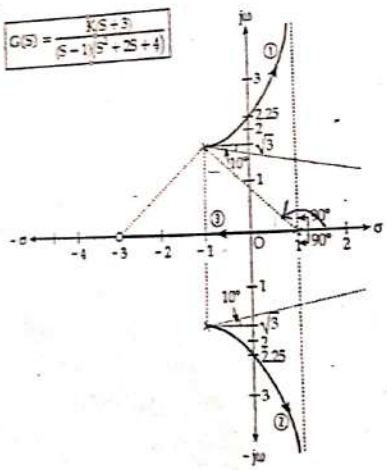
Step 1

Open loop poles at $S_1 = 1, S_2 = -1 + j\sqrt{3} = -1 + j1.73$ and $S_3 = -1 - j\sqrt{3} = -1 - j1.73$

Open loop zeros at $S_4 = -3$

Plotting these in graph;

$$G(S) = \frac{K(S+3)}{(S-1)(S^2+2S+4)}$$



Step 2

Number of open loop poles (P) = 3

Number of open loop zeros (Z) = 1

Number of root locus (N) = P = 3; since, (P > Z)

Step 3

Root locus lies between (1 to −1) and (−1 to −3).

i.e. Root locus lies between (1 to −3)

Step 4 Centroid of asymptotes

$$\sigma_A = \frac{\sum P - \sum Z}{P - Z} = \frac{1 + (-1 + j\sqrt{3}) + (-1 - j\sqrt{3}) - (-3)}{3 - 1} = \frac{2}{2} = 1$$

$\therefore$ Centroid lies at S = 1.

Step 5 Number of asymptotes

Number of asymptotes = P − Z = 3 − 1 = 2

Angle of asymptote $(\phi) = \frac{(2m+1) \times 180°}{P - Z}$

$$\phi_1 = \frac{(2 \times 0 + 1) \times 180°}{2} = 90° \text{ at } m = 0$$

$$\phi_2 = \frac{(2 \times 1 + 1) \times 180°}{2} = 270° \text{ at } m = 1$$

Step 6 Angle of departure for upper complex pole

Angle made by lower complex pole to this pole;

$\theta_1 = +90°$

Angle made by pole S = 1 to this pole;

$$\theta_2 = 90° + \tan^{-1}\left(\frac{2}{\sqrt{3}}\right) = 139.10°$$

Angle made by zeros S = −3 to this pole;

$$\theta_3 = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right) = 40.89°$$

$\therefore$ $\phi_d = 180° - (\theta_1 + \theta_2) + \theta_3 = 180° - (90° + 139.10°) + 40.89° = -8.2° \approx -9°$ (ↀ) i.e., 9° (ↀ)

Angle of departure for lower complex pole is just mirror of upper complex pole.

so, $\phi_d = +9°$ (ↀ)

Step 7 Break in point

The characteristics equation is;

$$1 + G(S)H(S) = 0$$

or, $$1 + \frac{K(S+3)}{(S-1)(S^2+2S+4)} = 0$$

or, $$(S-1)(S^2+2S+4) + K(S+3) = 0$$

or, $$K = -\left[\frac{(S-1)(S^2+2S+4)}{(S+3)}\right] = -\left[\frac{S^3+S^2+2S-4}{(S+3)}\right]$$

Perform $\frac{dK}{dS} = 0$; we have,

Scanned with CamScanner

$$\frac{dK}{dS} = -\left[\frac{2S^3 + 10S^2 + 6S + 10}{(S+3)^2}\right] = 0$$

or, $2S^3 + 10S^2 + 6S + 10 = 0$

or, $S^3 + 5S^2 + 3S + 5 = 0$

Solving; we get,

$S_1 = -4.58$ Invalid

$S_1 = -0.208 - j1.023$

$S_1 = -0.208 + j1.023$ } Complex; so not possible

Step 8 Jω crossing

Applying R-H criteria to characteristics equation;

$1 + G(S)H(S) = 0$

or, $(S-1)(S^2 + 2S + 4) + K(S+3) = 0$

or, $S^3 + S^2 + (2+K)S + (3K-4) = 0$

The R-H array becomes;

S^3 1 $(2+K)$

S^2 1 $(3K-4)$

S^1 b_1

$b_1 = -\frac{1}{1}\begin{vmatrix} 1 & (2+K) \\ 1 & (3K-4) \end{vmatrix} = -(3K-4-K-2) = 6-2K$

Therefore, the R - H array completes by;

S^3 1 $(2+K)$

S^2 1 $(3K-4)$

S^1 $(6-2K)$

S^0 $(3K-4)$

The only possible row to be zero is S^1 row.

$6 - 2K = 0$

or, $6 = 2K$

∴ $K = 3$; for marginal stability

For stability;

$6 - 2K > 0$ and, $3K - 4 > 0$

or, $6 > 2K$ or, $3K > 4$

∴ $K < 3$ ∴ $K > 1.33$

Hence, stable range of K is $(1.33 < K < 3)$.

Therefore, the auxiliary equation is;

$A(S) = 0$

or, $S^2 + (3K-4) = 0$

or, $S^2 + (3 \times 3 - 4) = 0$

or, $S^2 + 5 = 0$

or, $S^2 = -5$

or, $S^2 = (\pm j\sqrt{5})^2$

∴ $S = \pm j\sqrt{5} = \pm j2.24$

Therefore, root locus crosses Imaginary axis at $S = \pm j\sqrt{5}$ *i.e.*, at $S = \pm j2.24$. Now,

i) The system will be unstable when $K < 1.33$ and/or $K > 3$ *i.e.*, $(3 < K < 1.33)$.

ii) Value of gain parameter (K) at $S = -4.58$ can be obtained by applying magnitude criteria.

$|G(S)H(S)| = 1$

or, $\left|\frac{K(S+3)}{(S-1)(S^2+2S+4)}\right|_{S=-4.58} = 1$

or, $\frac{|K|\,|-4.58+3|}{|-4.58-1|\,|(-4.58)^2 + 2(-4.58) + 4|} = 1$

or, $\frac{K \times 1.58}{5.58 \times 15.8164} = 1$

∴ $K = 55.858$

Value of gain (K) at point $S = -4.58$

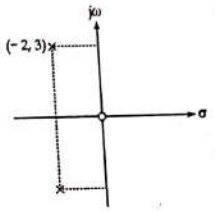
critically damped ($S_1 = S_2$)

iii) For over-damped condition; value of K should be small *i.e.*, $(x > K > 1.33)$. (real unequal.

For under-damped condition; value of K should be high *i.e.*, $(x < K < 1.33)$. (S_1 S_2 must be imaginary)

x is some constant value.

4. Draw root locus for the system that has open loop pole and zero plot in S-plane as shown in figure below. Also estimate the system gain at the point where the system exhibits critical damping. [2070 Chaitra]



Solution:

Step 1

Open loop poles at $S_1 = -2 + 3j$, $S_2 = -2 - 3j$

Open loop zero at $S_3 = 0$

Plotting these poles and zero on graph.

Step 2

Root locus lies in between (0 to −2) and (−2 to −∞) *i.e.*, segment between (0 to −∞) or entire L.H.P. is a part of a root locus.

Step 3

Number of open loop poles (P) = 2

Number of open zeros (Z) = 1 [∵ P > Z]

Number of root loci (N) = P = 2

Step 4 Centroid of asymptote (σ_A)

$$\sigma_A = \frac{\sum P - \sum Z}{P - Z} = \frac{(-2+3j) - 2 - 3j - 0}{2-1} = -4$$

Scanned with CamScanner

Step 5

Number of asymptote = $P - Z = 2 - 1 = 1$

Angle of asymptote $(\phi) = \frac{(2m+1)\times 180°}{P-Z}$

$\phi_1 = \frac{(2\times 0+1)\times 180°}{1} = 180°$ at $m = 0$

Step 6 Angle of departure (ϕ_d) for upper complex pole

Angle made by lower complex pole to upper $(\theta_1) = 90°$

Angle made by zero to upper pole $(\theta_2) = 90° + \tan^{-1}\left(\frac{2}{3}\right) = 90° + 33.69° = 123.69°$

Now,

$\phi_d = 180° - 90° + 123.69° = 213.69°(\circlearrowleft) = 146.31°(\circlearrowright)$

Similarly, angle of departure for lower complex pole is just mirror of upper complex pole.

so, $\phi_d = -213.69°(\circlearrowright) = 146.31°(\circlearrowleft)$

Step 7 Break in point

The characteristics equation is;

$1 + G(S)H(S) = 0$

From the question; we have,

$$G(S) = \frac{KS}{(S+2+3j)(S+2-3j)} = \frac{KS}{(S+2)^2+9} = \frac{KS}{S^2+4S+13}$$

so, $1 + G(S)H(S) = 0$

or, $1 + \frac{KS}{S^2+4S+13} = 0$

or, $S^2 + 4S + 13 + KS = 0$

or, $K = -\left(\frac{S^2+4S+13}{S}\right) = -(S + 4 + 13S^{-1})$

Performing, $\frac{dK}{dS} = 0$; we have,

$$\frac{dK}{dS} = -\left(1 - \frac{13}{S^2}\right) = 0$$

or, $S^2 - 13 = 0$

or, $S^2 = 13$

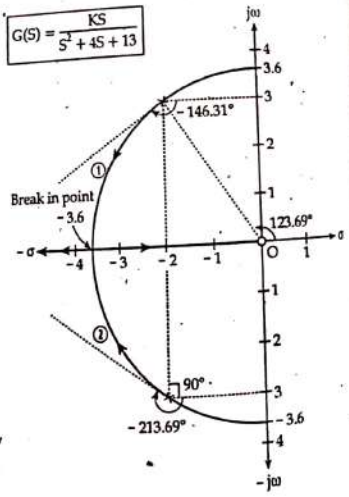
or, $S^2 = (\pm\sqrt{13})^2$

$\therefore$ $S_1 = \sqrt{13} = 3.605$ (Invalid)

and, $S_2 = -\sqrt{13} = -3.605$ (Valid)

Since, the root locus lies in entire L.H.P., so only $S = -3.605$ is the valid point and called break in point.

$\boxed{G(S) = \frac{KS}{S^2+4S+13}}$



Step 8 jω crossing

The characteristics equation is;

$S^2 + 4S + 13 + KS = 0$

or, $S^2 + (4 + K)S + 13 = 0$

The R-H array becomes;

S^2	1	13
S^1	$(K + 4)$	
S^0	13	

The only possible row to be zero is S^1 row.

When $K + 4 = 0$

$K = -4$

$\therefore$ The auxiliary equation is;

$A(S) = S^2 + 13$

Equating with zero; we have,

$S^2 + 13 = 0$

or, $S^2 = -13$

or, $S^2 = (\pm j\sqrt{13})^2$

$\therefore$ $S = \pm j3.60$

The root locus crosses the imaginary axis at $\pm j3.60$.

Step 9

The value of system gain (K) at critical damping

The value of K at break in point ($S = -3.605$) can be obtained by magnitude criteria;

$|G(S)H(S)| = 1$

or, $\left|\frac{KS}{S^2+4S+13}\right|_{S=-3.605} = 1$

or, $\frac{K\times 3.605}{(3.605)^2 - 4\times 3.605 + 13} = 1$

or, $K = \frac{11.58}{3.605} = 3.21$

$\therefore$ $K = 3.21$ at critical damping

5. Sketch the root locus for $G(S) = \frac{K}{S(S+6)(S^2+4S+13)}$ and derive the stability conditions. **[2071 Shrawan]**

Solution:

Step 1 Open loop poles

$S_1 = 0, S_2 = -6, S_3 = -2 + j3$ and $S_4 = -2 - j3$

Plot these poles on graph.

Step 2

Root locus lies in between (0 to −2) and (−2 to −6) i.e., entire range form (0 to −6)

Step 3

Number of open loop poles (P) = 4

Number of open loop zeros (Z) = 0

$\therefore$ Number of root locus (N) = P = 4; since, P > Z

Step 4 Centroid of asymptote

$$\sigma_A = \frac{\sum P - \sum Z}{P - Z} = \frac{-10 - 0}{4} = -2.5$$

Scanned with CamScanner

Step 5

Number of asymptote = $P - Z = 4 - 0 = 4$

Angle of asymptote $(\phi) = \frac{(2m+1)\times 180^\circ}{P-Z}$

so, $\phi_1 = \frac{(2\times 0+1)\times 180^\circ}{4} = 45^\circ$ at $m = 0$

$\phi_2 = \frac{(2\times 1+1)\times 180^\circ}{4} = 135^\circ$ at $m = 1$

$\phi_3 = \frac{(2\times 2+1)\times 180^\circ}{4} = 225^\circ$ at $m = 2$

$\phi_4 = \frac{(2\times 3+1)\times 180^\circ}{4} = 315^\circ$ at $m = 3$

Step 6 Angle of departure for upper complex pole

The angle made by pole R on pole P;

$\theta_1 = 90^\circ + \tan^{-1}\left(\frac{2}{3}\right) = 123.69^\circ$

The angle made by pole Q on pole P;

$\theta_2 = 90^\circ$

The angle made by pole S on pole P;

$\theta_3 = \tan^{-1}\left(\frac{3}{4}\right) = 36.87^\circ$

Therefore, angle of departure (ϕ_d) for upper complex pole is;

$\phi_{dP} = 180^\circ - (123.69^\circ + 90^\circ + 36.87^\circ) = -70.56^\circ$ (ʊ) or 70.56° (ʊ)

The angle of departure for lower complex pole Q is just mirror of upper complex pole.

$\therefore \quad \phi_{dQ} = 70.56^\circ$ (ʊ)

Step 7 Breakaway point

The characteristics equation is;

$1 + G(S)H(S) = 0$

or, $1 + \frac{K}{S(S+6)(S^2+4S+13)} = 0$

or, $S(S+6)(S^2+4S+13) + K = 0$

or, $S^4 + 10S^3 + 37S^2 + 78S + K = 0$

Perform $\frac{dK}{dS} = 0$; we have,

$4S^3 + 30S^2 + 74S + 78 = 0$

Solving; we get,

$S_1 = -4.20$ Valid

$S_2 = -1.65 - j1.39$, $S_3 = -1.65 + j1.39$ } Complex and not possible

Step 8 Jω crossing

The characteristics equation is;

$S^4 + 10S^3 + 37S^2 + 78S + K = 0$

Applying R-H criteria; we have,

$$G(S) = \frac{K}{S(S+6)(S^2+4S+13)}$$

S^4	1	37	K
S^3	10	78	0
S^2	b_1	b_2	

$b_1 = -\frac{1}{10}\begin{vmatrix} 1 & 37 \\ 10 & 78 \end{vmatrix} = 29.2$

$b_2 = -\frac{1}{10}\begin{vmatrix} 1 & K \\ 10 & 0 \end{vmatrix} = K$

Similarly,

S^4	1	37	K
S^3	10	78	0
S^2	29.2	K	
S^1	C_1	C_2	
S^0	d_1		

$C_1 = -\frac{1}{29.2}\begin{vmatrix} 10 & 78 \\ 29.2 & K \end{vmatrix} = \frac{2277.6 - 10K}{29.2}$

$C_2 = 0$

and, d_1 = Constant term = K

Now, the complete R-H array becomes;

S^4	1	37	K
S^3	10	78	0
S^2	29.2	K	
S^1	$\left(\frac{2277.6-10K}{29.2}\right)$	C_2	
S^0	K		

For stability;

$K > 0$

and, $\frac{2277.6 - 10K}{29.2} > 0$

or, $227.76 > K$

For stability $(0 < K < 227.76)$

and, marginal stability;

$\frac{2277.6 - 10K}{29.2} = 0$

$\therefore \quad K = 227.76$

The auxiliary equation is;

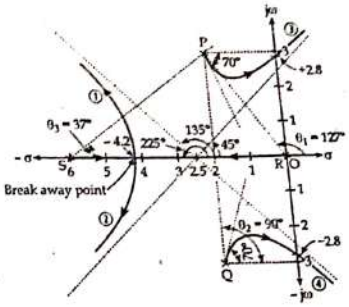
$A(S) = 29.2S^2 + 227.76 = 0$

or, $29.2S^2 + 227.76 = 0$

or, $S^2 = -7.8$

or, $S = (\pm j2.79)$

i.e., the root locus cross imaginary axis at $S = \pm j2.8$.



Scanned with CamScanner

6. **Construct a root locus for unity feedback system with OLTF; $G(S) = \frac{K(S+2)}{S(S-1)}$. Hence, show that the system is unstable for small value of gain and over damped at larger value of gain.** [2072 Kartik]

Solution:

Step 1

Open loop poles at $S_1 = 0, S_2 = 1$

Open loop zeros at $S_3 = -2$

Plot these points on graph.

Step 2

Root locus lies in between (0 to 1) and (−2 to −∞).

Step 3

Number of open loop poles (P) = 2

Number of open loop zero (Z) = 1

∴ Number of root loci (N) = P = 2; since, P > Z

Step 4 Centroid of asymptote

$$\sigma_A = \frac{\sum P - \sum Z}{P - Z} = \frac{(0+1)+(-2)}{2-1} = 3$$

Step 5

Number of asymptote = 2 − 1 = 1

Angle of asymptote $(\phi) = \frac{(2m+1)\times 180^\circ}{P-Z}$

$$\phi_1 = \frac{(2\times 0+1)\times 180^\circ}{2-1} = 180^\circ \text{ at } m = 0$$

Step 6 Break in and breakaway point

The characteristics equation is;

$$1 + G(S)H(S) = 0$$

or, $1 + \frac{K(S+2)}{S(S-1)} = 0$

or, $(S^2 - S) + K(S+2) = 0$

or, $K = -\frac{(S^2 - S)}{(S+2)}$

Perform $\frac{dK}{dS} = 0$; we have,

$$\frac{dK}{dS} = -\left[\frac{S^2 + 4S - 2}{(S+2)^2}\right] = 0$$

Solving; we get,

$$S^2 + 4S - 2 = 0$$

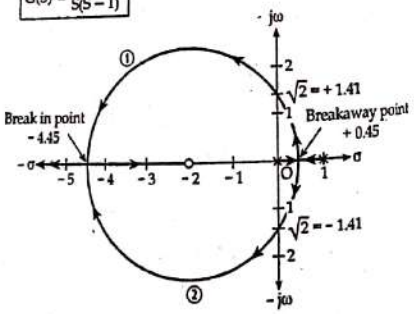
Using calculator; we have,

$S_1 = -2 + \sqrt{6} = 0.45$ Valid

and, $S_2 = -2 - \sqrt{6} = -4.45$ Valid

Since, root locus lies in between (0 to 1) and (−2 to −∞) both points are valid.

$G(S) = \frac{K(S+2)}{S(S-1)}$



Step 7 jω crossing

The characteristics equation is;

$$(S^2 - S) + K(S+2) = 0$$

or, $S^2 + S(K-1) + 2K = 0$

Now, applying R-H criteria;

S^2	1	2K
S^1	(K − 1)	
S^0	2K	

For marginal stability, the only row to be zero is S^1 row;

so, $K - 1 = 0$

∴ $K = 1$

and, for stability; we have,

$K > 1$

and, $K > 0$ as well *i.e.*, for stability $(\infty > K > 1)$

Now, the auxiliary equation is;

$A(S) = 0$

or, $S^2 + 2K = 0$

or, $S^2 + 2 \times 1 = 0$

or, $S^2 = -2$

or, $S^2 = (\pm j\sqrt{2})^2$

or, $S = \pm j\sqrt{2}$

∴ $S = \pm j1.41$

The root locus crosses jω axis at $S = \pm j1.41$.

The value of K at breakaway point S = 0.45

Magnitude criteria;

$$|G(S)H(S)| = 1$$

or, $\left|\frac{K(S+2)}{S(S-1)}\right|_{S=0.45} = 1$

or, $\frac{K \times 2.45}{0.45 \times 0.55} = 1$

and, $K = 0.10$

The value of K at break in point S = −4.45

$$\left|\frac{K(S+2)}{S(S-1)}\right|_{S=-4.45} = 1$$

and, $K = 9.9$

At S = 0.45;

$K = 0.10$

At S = −4.45;

$K = 9.9$

For over-damped system; the roots must be real and unequal *i.e.*, $(K > 9.9)$ and also $(0 < K < 0.10$

For under-damped system; the roots are complex conjugate. The range of K is $(0.10 < K < 9.9)$.

And at K = 0.10 and K = 9.9; roots are real and equal. Hence, it is critically damped.

Scanned with CamScanner

7. A unity feedback system has an open loop transfer function $G(S) = \frac{K(S+9)}{S(S^2+4S+11)}$. Sketch root locus and determine: [2073 Shrawan]
   i) range of K for system to be stable
   ii) undamped natural frequency of oscillation

Solution:

Step 1

Open loop poles at $S_1 = 0, S_2 = -2+\sqrt{7}i, S_3 = -2-\sqrt{7}i$

Open loop zeros at $S_4 = -9$

Now, plot these points on graph.

Step 2

Root locus lies in between (0 to -2) and (-2 to -9)

Step 3

Number of open loop poles (P) = 3

Number of open loop zeros (Z) = 1

$\therefore$ Number of root locus (N) = P = 3; since, P > Z

Step 4 Centroid of asymptotes

$$\sigma_A = \frac{\sum P - \sum Z}{P-Z} = \frac{-4-(-9)}{3-1} = \frac{5}{2} = 2.5$$

Step 5

Number of asymptote = P − Z = 3 − 1 = 2

Angle of asymptote $(\phi) = \frac{(2m+1)\times 180^\circ}{P-Z}$

$$\phi_1 = \frac{(2\times 0+1)\times 180^\circ}{3-1} = 90^\circ \text{ at } m = 0$$

$$\phi_2 = \frac{(2\times 1+1)\times 180^\circ}{3-1} = 270^\circ \text{ at } m = 1$$

Step 6 Breakaway point

The characteristics equation is;

$$1 + G(S)H(S) = 0$$

or, $$1 + \frac{K(S+9)}{S(S^2+4S+11)} = 0$$

or, $$S(S^2+4S+11) + K(S+9) = 0$$

or, $$S^3 + 4S^2 + (11+K)S + 9K = 0$$

or, $$K = -\left(\frac{S^3+4S^2+11S}{S+9}\right)$$

Perform $\frac{dK}{dS} = 0$

or, $$-\left[\frac{2S^3+31S^2+72S+99}{(S+9)^2}\right] = 0$$

or, $$-(2S^3+31S^2+72S+99) = 0$$

or, $$2S^3+31S^2+72S+99 = 0$$

Solving using calculator; we have,

$S_1 = -13.03$ (Invalid)

$S_2 = -1.236 + j1.5$ (Invalid)

$S_3 = -1.236 - j1.5$ (Invalid)

S_2 and S_3 are complex and invalid.

Step 7 Angle of departure for upper complex pole

Angle made by pole R on pole P;

$$\theta_1 = 90^\circ + \tan^{-1}\left(\frac{2}{\sqrt{7}}\right) = 127^\circ$$

Angle made by pole Q on pole P;

$$\theta_2 = +90^\circ$$

Angle made by zero S on pole P;

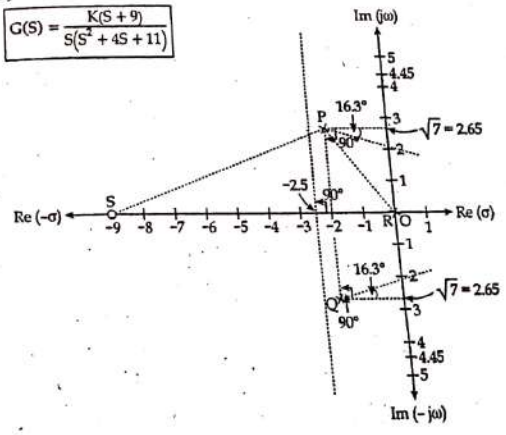
$$\theta_3 = \tan^{-1}\left(\frac{\sqrt{7}}{7}\right) = +20.7^\circ$$

Therefore, angle of departure for upper complex pole P is;

$$\phi_{dQ} = 180^\circ - (127^\circ + 90^\circ) + 20.7^\circ = -16.3^\circ = 16.3^\circ (\text{U}) = 343.7^\circ (\text{O})$$

The angle of departure for lower complex pole Q is just mirror of upper complex pole.

$\therefore$ $\phi_{dP} = 16.3^\circ$ (O)



Step 8 jω crossing

The characteristics equation is;

$$1 + G(S)H(S) = 0$$

or, $$S^3 + 4S^2 + (11+K)S + 9K = 0$$

Now, R-H array becomes;

S^3	1	$(11+K)$
S^2	4	$9K$
S^1	b_1	
S^0	C_1	

Scanned with CamScanner

$$b_1 = -\frac{1}{4}\begin{vmatrix} 1 & (11+K) \\ 4 & 9K \end{vmatrix} = \frac{4(11+K) - 9K}{4} = \left(\frac{44-5K}{4}\right)$$

and, $C_1 = 9K$ = Constant term of characteristics equation

Now, the array becomes;

S^3	1	$(11+K)$
S^2	4	$9K$
S^1	$\left(\frac{44-5K}{4}\right)$	
S^0	$9K$	

The S^0 row cannot be zero. So, only possible row to be zero is S^1 row.

i.e., $\left(\frac{44-5K}{4}\right) = 0$

or, $44 - 5K = 0$

or, $K = \frac{44}{5}$

The system will be stable; when,

$9K > 0$ and, $\left(\frac{44-5K}{4}\right) > 0$

i.e., $K > 0$ or, $(44 - 5K) = 0$

$\therefore \left(\frac{44}{5}\right) > K$

$\therefore$ Range of K for the system to be stable is $\boxed{\left(\frac{44}{5} > K > 0\right)}$.

Now, the auxiliary equation is;

$4S^2 + 9K = 0$

or, $4S^2 + 9\left(\frac{44}{5}\right) = 0$

or, $20S^2 + 396 = 0$

or, $S^2 = -\frac{396}{20} = -\frac{99}{5}$

$\therefore S_{1,2} = \pm j4.45$

The root locus intersects jω axis on ± j4.45.

$\therefore$ Undamped frequency of oscillation $\boxed{\omega = 4.45 \text{ rad/sec.}}$

Scanned with CamScanner

Chapter 6

Frequency Response Techniques

6.1 Frequency domain characterization of the system 172
6.2 Relationship between real and complex frequency response 172
6.2.1 Frequency domain specification 173
6.3 Bode plot: magnitude and phase plot 173
6.3.1 Bode plot of some standard form 174
6.3.2 Resonant frequency and resonant peak for second order system 178
6.3.3 Initial slope, initial magnitude and intersection with $\log_{10} \omega$ (*i.e.*, with x – axis) 179
6.3.4 General rule of construction of Bode-plot 180
6.3.5 Minimum phase system and non-minimum phase systems 189
6.4 Effect of gain and time constant on Bode plot 189
6.4.1 Effect of gain (K) 189
6.4.2 Effect of time constant 189
6.5 Phase margin, gain margin, phase crossover frequency, gain crossover frequency and stability from Bode diagram 190
6.6 Polar plot and stability 191
6.6.1 Polar plot 191
6.6.2 Procedure to sketch polar plot 191
6.6.3 Polar plots of some standard type functions 192
6.6.4 Polar plot of type zero system 193
6.6.5 Polar plot of type I system 196
6.6.6 Polar plot of type –II system 198
6.6.7 Effect of additional pole on type –II system 198
6.6.8 Polar plot involving quadratic factor 200
6.6.9 Stability analysis from polar plot 201
6.7 Nyquist plot and stability 206
6.7.1 Mapping theory 207
6.7.2 Nyquist path, Nyquist contour and Nyquist plot 207
6.7.3 Stability criteria 208
6.7.4 Steps to sketch Nyquist plot 208

Frequency response means the steady state response of a system subjected to sinusoidal input *i.e.*, we vary the frequency of input signal over certain range and study the resulting output. In other word, the magnitude and phase relationship between sinusoidal input and steady state output of a system is known as frequency response.

Scanned with CamScanner

The frequency response analysis is done by following methods:

i) Bode plot
ii) Polar plot
iii) Nyquist plot

6.1 FREQUENCY DOMAIN CHARACTERIZATION OF THE SYSTEM

If the input signal;

$$r(t) = X \sin \omega t$$

The output can be written as;

$$C(t) = Y \sin(\omega t + \phi)$$

From control system aspect;

$$G(S) = \frac{C(S)}{R(S)}$$

If $S = j\omega$ (oscillatory system); then, $G(j\omega)$ is sinusoidal transfer function.

Put, $S = j\omega$

$$G(j\omega) = \frac{C(j\omega)}{R(j\omega)}$$

i.e., $C(j\omega) = G(j\omega) \times R(j\omega)$

For any input, to understand the nature of output we have to study $G(j\omega)$.

For stable system, the sinusoidal output is of the same frequency as input but the amplitude and phase of the output will be different from input.

$G(j\omega)$ is a complex quantity;

$$G(j\omega) = G_x + jG_y = |G|\angle\phi = M\angle\phi$$

where, $M = \frac{\text{Amplitude of output}}{\text{Amplitude of input}}$ = Gain or magnitude ratio

$$\phi = \tan^{-1}\left(\frac{G_y}{G_x}\right) = \text{Phase difference between output and input}$$

Advantages of frequency domain analysis

i) If transfer function is difficult to be obtained by analytical methods, the frequency response of stable open loop system can be obtained experimentally.
ii) Simple and accurate

Disadvantages of frequency domain analysis

i) Obtaining frequency response practically is time consuming.
ii) For high time constant, frequency response method is not convenient.

6.2 RELATIONSHIP BETWEEN REAL AND COMPLEX FREQUENCY RESPONSE

Effect of adding a pole and zero

When a pole is added in the forward path for a second order system:

i) Bandwidth increases
ii) Rise time (t_r) increases
iii) Resonant peak (M_r) increases
iv) System becomes less stable

When a zero is added in the forward path:

i) Bandwidth increases
ii) Less rise time (t_r)
iii) Settling time (t_s) increases
iv) System becomes stable

6.2.1 Frequency Domain Specification

i) Resonant peak (M_r)

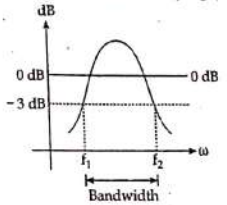
The maximum value of magnitude is known as resonant peak. The magnitude of resonant peak gives the information about the relative stability of the system. A large value of resonant peak implies undesirable transient response.

ii) Resonant frequency (ω_r)

The frequency at which magnitude has maximum value is known as resonant frequency. If ω_r is large, the time response is fast.

iii) Bandwidth (BW)

Range of frequency in which the magnitude of close loop does not drop - 3 dB. In other words we can say that the bandwidth is defined as the band of frequencies lying between -3 dB points.



$\therefore$ Bandwidth (BW) $= |f_1 - f_2| = |f_2 - f_1|$

iv) Cut-off frequency (ω_b)

The frequency at which magnitude is 3 dB below its zero frequency value is called cut off frequency (ω_b).

v) Cut-off rate

The cut off rate is the slope of the log magnitude curve near the cut off frequency.

6.3 BODE PLOT: MAGNITUDE AND PHASE PLOT

It is common method of representing sinusoidal Transfer function of the given system graphically, for determining the stability. It consists of two separate parts:

i) Bode Magnitude plot

Let a transfer function be $G(S) = \frac{K(1+ST_a)(1+ST_b)\ldots\ldots}{S^N(1+ST_1)(1+ST_2)\ldots\ldots}$.

The graphical representation between $20\log_{10}|G(j\omega)|$ in y-axis vs $\log_{10}\omega$ in x-axis is magnitude plot.

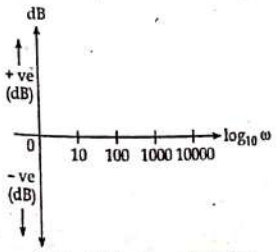


Figure: (i) Bode magnitude plot

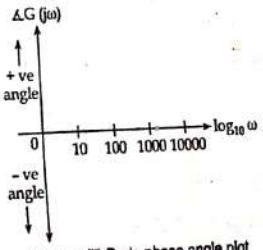


Figure: (ii) Bode phase angle plot

Scanned with CamScanner

Put, $S = j\omega$ on transfer function; we have,

$$G(j\omega) = \frac{K(1+j\omega T_a)(1+j\omega T_b)\ldots\ldots}{(j\omega)^N(1+j\omega T_1)(1+j\omega T_2)\ldots\ldots}$$

Taking magnitude; we have,

$$|G(j\omega)| = \frac{K\sqrt{1+\omega^2T_a^2}\sqrt{1+\omega^2T_b^2}}{\omega^N\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}}$$

Taking $\log_{10}$ on both sides; we have,

$$20\log_{10}|G(j\omega)| = 20\log_{10}K + 20\log_{10}\sqrt{1+\omega^2T_a^2} + 20\log_{10}\sqrt{1+\omega^2T_b^2} - 20N\log_{10}\omega - 20\log_{10}\sqrt{1+\omega^2T_1^2} - 20\log_{10}\sqrt{1+\omega^2T_2^2}$$

ii) Bode-phase plot

The plot of $\angle G(j\omega)$ in Y-axis and $\log_{10}\omega$ frequency in x-axis is known as Bode-phase plot.

$$\angle G(j\omega) = \tan^{-1}(\omega T_a) + \tan^{-1}(\omega T_b) - N\times 90° - \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

Advantages of Bode Plot

i) Easiest method in frequency response technique.
ii) Multiplication and division of magnitude can be converted into addition and subtraction respectively.
iii) Too low and too high frequency response can be study on a single graph.

6.3.1 Bode Plot of Some Standard Form

i) The gain K

For gain (K);

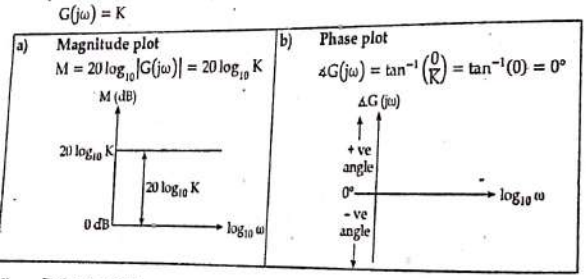
Let, $G(S) = K$

and, put $S = j\omega$

$$G(j\omega) = K$$

a) **Magnitude plot**

$$M = 20\log_{10}|G(j\omega)| = 20\log_{10}K$$



b) **Phase plot**

$$\angle G(j\omega) = \tan^{-1}\left(\frac{0}{K}\right) = \tan^{-1}(0) = 0°$$

ii) Pole at origin

$$G(S) = \frac{1}{S^N}$$

Put, $S = j\omega$; then,

$$G(j\omega) = \frac{1}{(j\omega)^N}$$

a) **Magnitude plot**

$$M = 20\log_{10}|G(j\omega)| = 20\log_{10}\left|\frac{1}{(j\omega)^N}\right| = 20\log_{10}\left[\frac{1}{\omega^N}\right] = 20\log_{10}(\omega^{-N}) = -20N\log_{10}\omega$$

The plot of [illegible]

For N = 1; the line has a slope of −20 dB/decade

For N = 2; the line has a slope of −40 dB/decade and so on.

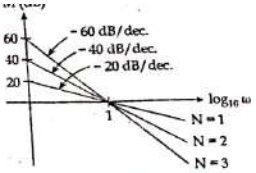


Figure: Magnitude plot

b) **Phase plot**

For $G(j\omega) = \frac{1}{(j\omega)^N} = (j\omega)^{-N}$

$$\angle G(j\omega) = -(90\times N)°$$

where, N = 1, 2, 3, ...

Slope of magnitude plot $= \left(\frac{Y}{X}\right) = \frac{-20N\log_{10}\omega}{\log_{10}\omega}$

$= -20N$ dB/decade

For N = 1; the angle is −90°.

For N = 2; the angle is −180°.

i.e., When N = 1;

Slope = −20 dB/decade

Phase = −90°

When N = 2;

Slope = −40 dB/decade

Phase = −180° and so on

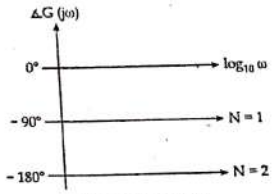


Figure: Phase plot

iii) Zeros at origin

Let, $G(S) = S$

Put, $S = j\omega$; then,

$$G(j\omega) = j\omega$$

a) **Magnitude plot**

$$|G(j\omega)| = |j\omega| = \omega$$

Magnitude (M) $= 20\log_{10}|G(j\omega)| = 20\log_{10}\omega$

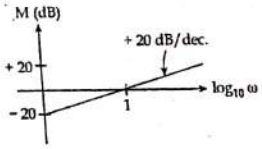


Figure: Magnitude plot

b) **Phase plot**

$$\angle G(j\omega) = \tan^{-1}\left(\frac{\omega}{0}\right) = \tan^{-1}(\infty) = '+90°$$

The plot (M) vs $\log_{10}\omega$ is a straight line having a lope of +90° +20 dB/decade and angular phase shift of +90°.

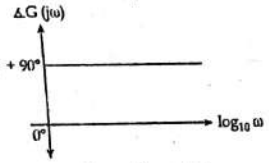


Figure: Phase plot

iv) $(1+ST)^N$ in zero

Here,

$$G(S) = (1+ST)^N$$

Put, $S = j\omega$

$$G(j\omega) = (1+j\omega T)^N$$

$$|G(j\omega)| = \left(\sqrt{1+\omega^2T^2}\right)^N$$

a) **Magnitude plot**

Here,

$$M = 20\log_{10}|G(j\omega)| = 20\log_{10}\left[\left(\sqrt{1+\omega^2T^2}\right)^N\right] = 20N\log_{10}\left(\sqrt{1+\omega^2T^2}\right)$$

Scanned with CamScanner

i) When $\omega T \ll 1$; then,

$$M = 20N \log_{10}\sqrt{1} = 20N \log_{10}(1) = 0 \text{ dB}$$

ii) When $\omega T \gg 1$; then,

$$M = 20N \log_{10}(\omega T) = 20N \log_{10}(\omega) + 20N \log_{10}(T) = 20N \log_{10}(\omega) - 20N \log_{10}\left(\frac{1}{T}\right)$$

Equating the above equation with magnitude zero;

i.e., $20N \log_{10}(\omega) - 20N \log_{10}\left(\frac{1}{T}\right) = 0$

We get,

$\boxed{\omega = \frac{1}{T}}$ which is called corner frequency or break frequency.

Corner frequency is that frequency where magnitude plot change its slope.

[Since, y = mx + C; where, m is the slope.]

Slope = + 20N dB/decade

For N = 1;

Slope = + 20 dB/decade

For N = 2;

Slope = + 40 dB/decade

For N = 3;

Slope + 60 dB/decade and so on.

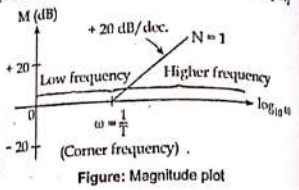


Figure: Magnitude plot

Thus, the two parts of the graph intersects the 0 dB axis at $\omega = \frac{1}{T}$.

The second part is a straight line having the slope of + 20 dB/decade.

b) Phase angle plot

$$\angle G(j\omega) = N \times \tan^{-1}(\omega T)$$

1. At very low frequencies; (ωT) is very small.
   so, $\angle G(j\omega) = N \times \tan^{-1}(0) = 0°$
2. At $\omega T = 1$;
   $\angle G(j\omega) = N \times \tan^{-1}(1) = N \times 45° = (45N)°$
3. At high frequencies;
   $\angle G(j\omega) = N \times \tan^{-1}(\infty) = N \times 90° = (90N)°$

Thus, the value of $\angle G(j\omega)$ gradually changes from 0° to 90° × N as ω increase from 0 to very high value.

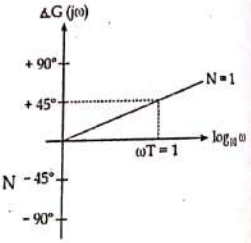


Figure: Phase plot

$(1 + ST)^N$ at poles

$$G(S) = \frac{1}{(1 + ST)^N}$$

Put, $S = j\omega$

$$G(j\omega) = \frac{1}{(1 + j\omega T)^N}$$

$$|G(j\omega)| = \frac{1}{\left(\sqrt{1 + \omega^2 T^2}\right)^N}$$

a) Magnitude plot

$$M = 20 \log_{10}|G(j\omega)| = 20 \log_{10}\left(+\sqrt{1 + \omega^2 T^2}\right)^{-N} = -20N \log_{10}\sqrt{1 + \omega^2 T^2}$$

i) When $\omega T \ll 1$ (at lower frequencies)

$$M = -20N \log_{10}(1) = 0$$

ii) When $\omega T \gg 1$ (at higher frequencies)

$$M = -20N \log_{10}(\omega T)$$

$$= -20N \log_{10}\omega - 20N \log_{10} T$$

$$= -20N \log_{10}\omega + 20N \log_{10}\left(\frac{1}{T}\right)$$

Here, M vs $\log_{10}\omega$ has two parts.

Comparing with y = mx + c; we get,

Slope (m) = −20 N dB/dec.

and, N = 1; Slope = −20 dB/dec.

N = 2; Slope = −40 dB/dec.

N = 3; Slope = −60 dB/dec.

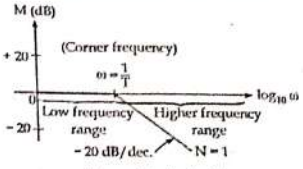


Figure: Magnitude plot

To determine corner frequency, M = 0

so, $-20N \log_{10}\omega - 20N \log_{10}\left(\frac{1}{T}\right) = 0$

or, $-\log_{10}\omega = -\log_{10}\left(\frac{1}{T}\right)$

$\therefore \quad \omega = \frac{1}{T}$; which is called break frequency or corner frequency.

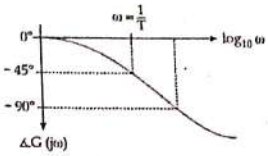


Figure: Phase plot

b) Phase plot

$$\angle G(j\omega) = -N \times \tan^{-1}(\omega T)$$

1. At very low frequencies
   $\angle G(j\omega) = -N \times \tan^{-1}(0) = 0°$
2. At corner frequency, ($\omega T = 1$)
   $\angle G(j\omega) = -N \times \tan^{-1}(1) = -N \times 45° = -(45\ N)°$
3. At higher frequency;
   $\angle G(j\omega) = -N \times \tan^{-1}(\infty) = -N \times 90° = -(90\ N)°$

vi) Second order system

Let, $G(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$

Put, $S = j\omega$;

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + j2\zeta\omega\omega_n} = \frac{\omega_n^2}{\omega_n^2\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + \omega_n^2 j2\zeta\left(\frac{\omega}{\omega_n}\right)}$$

$$= \frac{1}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + j2\zeta\left(\frac{\omega}{\omega_n}\right)}$$

a) Magnitude plot

Now,

$$M = 20 \log_{10}|G(j\omega)| = 20 \log_{10}\left|\frac{1}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + j2\zeta\left(\frac{\omega}{\omega_n}\right)}\right|$$

$$= -20 \log_{10}\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}$$

Scanned with CamScanner

Let, $u = \frac{\omega}{\omega_n}$; then,

$$M = -20\log_{10}\sqrt{(1-u^2)^2 + 4\zeta^2u^2}$$

Consider two cases:

1. When $u \ll 1$ *i.e.*, $\frac{\omega}{\omega_n} \ll 1$

$$M = -20\log_{10} 1 = 0\ \text{dB}$$

2. When $u \gg 1$ *i.e.*, $\frac{\omega}{\omega_n} \gg 1$

$$M = -20\log_{10}\sqrt{(u^2)^2} = -20\log_{10} u^2 = -40\log_{10} u$$

i.e., straight line having slope of −40 dB/dec. and passing through point u.

Therefore, the plot consists of;

i) $M = 0$ at $u \ll 1$

ii) $M = -40\log_{10} u$ at $u \gg 1$

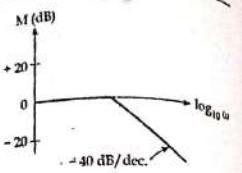


Figure: Magnitude plot

b) Phase plot

Now,

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right)$$

1. For small value of u, u^2 is very small

$$\therefore \quad \angle G(j\omega) = -\tan^{-1}(2\zeta u)$$

2. For large value of u, $u \gg 1$

$$\angle G(j\omega) = +\tan^{-1}\left(\frac{2\zeta}{u}\right)$$

3. When $u = 1$; then,

$$\angle G(j\omega) = -\tan^{-1}(\infty) = -90°$$

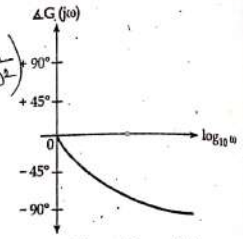


Figure: Phase plot

6.3.2 Resonant Frequency and Resonant Peak for Second Order System

For second order system

$$M = 20\log_{10}\left[\frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}}\right]$$

Consider,

$$g(\omega) = \left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2$$

For M to be maximum, $g(\omega)$ should be minimum.

i.e., $\frac{dg(\omega)}{d\omega} = 0$

$$\frac{dg(\omega)}{d\omega} = 2\times\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]\left(-\frac{2\omega}{\omega_n^2}\right) + 4\zeta^2\times 2\left(\frac{\omega}{\omega_n^2}\right) = 0$$

$$-4\left(\frac{\omega}{\omega_n^2}\right)\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right] + 8\zeta^2\left(\frac{\omega}{\omega_n^2}\right) = 0$$

$$\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right] - 2\zeta^2 = 0$$

$$\left[1-\frac{\omega^2}{\omega_n^2}\right] = 2\zeta^2$$

or, $\omega_n^2 - \omega^2 = 2\zeta^2\omega_n^2$

or, $\omega_n^2(1-2\zeta^2) = \omega^2$

$\therefore \quad \omega = \omega_n\sqrt{1-2\zeta^2}$

This is the resonant frequency in Bode magnitude plot.

and, Magnitude $(M_{Resonance}) = 20\log_{10}\left[\frac{1}{\sqrt{\left[1-(1-2\zeta^2)\right]^2 + 4\zeta^2(1-2\zeta^2)}}\right]$

$$= 20\log_{10}\left(\frac{1}{\sqrt{4\zeta^4 + 4\zeta^2 - 8\zeta^4}}\right) = 20\log_{10}\left(\frac{1}{\sqrt{4\zeta^2 - 4\zeta^4}}\right) = 20\log_{10}\left(\frac{1}{2\zeta\sqrt{1-\zeta^2}}\right)$$

$\therefore \quad \boxed{M_{Resonance}\ (Peak) = 20\log_{10}\left(\frac{1}{2\zeta\sqrt{1-\zeta^2}}\right)}$

More the value of ζ, smaller the value of resonant peak and vice-versa.

For no resonance;

$$M_{Resonance}\ (Peak) = 0$$

i.e., $\log_{10}\left(\frac{1}{2\zeta\sqrt{1-\zeta^2}}\right) = 0$

or, $\frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1 \qquad [\because \log_{10} 1 = 0]$

or, $1 = 2\zeta\sqrt{1-\zeta^2}$

Squaring both sides; we have,

$$1 = 4\zeta^2(1-\zeta^2)$$

or, $4\zeta^4 - 4\zeta^2 + 1 = 0$

or, $(2\zeta^2 - 1)^2 = 0$

or, $2\zeta^2 = 1$

$\therefore \quad \boxed{\zeta = \frac{1}{\sqrt{2}}}$; for no resonance

6.3.3 Initial Slope, Initial Magnitude and Intersection with $\log_{10}\omega$ (i.e., with x-axis)

Let, $G(S)H(S) = \frac{K}{S^N}$

For unity feedback system;

$$H(S) = 1$$

and, put, $S = j\omega$;

$$G(j\omega) = \frac{K}{(j\omega)^N}$$

or, $|G(j\omega)| = \frac{|K|}{|\omega^N|}$

Taking magnitude; we have,

$$M = 20\log_{10}|G(j\omega)| = 20\log_{10}(K) - 20N\log_{10}\omega$$

i.e., $\boxed{M = 20\log_{10} K - 20N\log_{10}\omega}$

Comparing with $y = mx + C$; we have,

$\boxed{\text{Slope} = -20\ N\ \text{dB/dec.}}$

Scanned with CamScanner

To cross the $\log_{10}\omega$ axis (or x-axis);

Magnitude = 0 dB

or, $20\log_{10}K = 20N\log_{10}\omega$

or, $K = \omega^N$

and, $\boxed{\omega = (K)^{\frac{1}{N}}}$

When N = 0; Type zero system; Here, $K = K_p$

N = 1; Type one system; Here, $K = K_v$

N = 2; Type two system; Here, $K = K_a$

Therefore,

Type of system (N)	Initial slope	Intersection with Parallel to x-axis
0	0 dB/dec	k
1	− 20 dB/dec	$\sqrt{K}$
2	− 40 dB/dec	$(K)^{\frac{1}{3}}$
3	− 60 dB/dec	⋮
⋮	⋮	$(K)^{\frac{1}{N}}$
N	− 20 N dB/dec	

6.3.4 General Rule for Construction of Bode–Plot

Rule 1

Express transfer function in the form of,

$$G(S) = \frac{K(1+ST_a)(1+ST_b)\cdots\cdots}{S^N(1+ST_1)(1+ST_2)\left[1+2\zeta\left(\frac{\omega}{\omega_n}\right)+\left(j\frac{\omega}{\omega_n}\right)^2\right]\cdots\cdots}$$

Rule 2

Put S = jω and identify corner frequency,

i) Corner frequency for zeros

$$\left(\frac{1}{T_a}\right), \left(\frac{1}{T_b}\right), \cdots\cdots$$

ii) Corner frequency for poles

$$\left(\frac{1}{T_1}\right), \left(\frac{1}{T_2}\right), \cdots\cdots, \omega_n$$

↑ For second degree

Choose lowest corner frequency (either from pole or zeros) and perform $= \frac{1}{10} \times$ Smallest corner frequency

Approximate this with nearest decade scale, this is ω.

Rule 3 Identify type N system

i) Starting magnitude in dB $= 20\log_{10}K - 20N\log_{10}\omega$

where, ω is the smallest decade frequency.

N is the type of system.

ii) Starting slope

a) For type 0 system (N = 0);

Initial slope = 0 dB/dec.

b) For type 1 system (N = 1);

Initial slope = −20 dB/dec.

c) For type 2 system (N = 2);

Initial slope = −40 dB/dec. and so on

Rule 4

From starting magnitude to initial corner frequencies (smallest), draw line with initial slope guided by type N system and from that frequency add contribution of poles and zeros on slope and so on.

Rule 5

Calculate individual phase angle contributed by each factors (pole and zeros) at different frequency and Determine total phase angle (Φ_{Total}).

EXAMPLE 6.1

Sketch the Bode plot of closed loop unity feedback system with open loop transfer function (OLTF) as $G(S) = \frac{1000}{S(1+0.1S)(1+0.001S)}$ **[2067 Magh]**

Solution:

Step 1

The transfer function is written in the form of $G(S) = \frac{1000}{S(1+0.1S)(1+0.001S)}$.

Here,

$K = 1000$

Step 2

Put, S = jω; then,

$$G(j\omega) = \frac{1000}{j\omega(1+0.1j\omega)(1+0.001j\omega)}$$

Step 3 Determining corner frequencies

Corner frequency for poles;

$\omega_1 = \frac{1}{0.1} = 10$ rad/sec. (lowest)

$\omega_2 = \frac{1}{0.001} = 1000$ rad/sec.

Now,

Starting frequency $(\omega) = \frac{1}{10} \times$ Lowest corner frequency $= \frac{1}{10} \times 10 = 1$ rad/sec.

Step 4 Magnitude plot

i) Starting magnitude

$M = 20\log_{10}K - 20N\log_{10}\omega$

Here,

N = 1 = Type one system

so, $M = 20\log_{10}(1000) - 20\log_{10}(1) = 60 - 0 = 60$ dB

ii) From ω = 1 rad/sec. to ω = 10 rad/sec.

Slope = −20 dB/dec. (due to type-first)

From ω = 10 rad/sec. to ω = 1000 rad/sec.

Slope = −20 + (−20) (due to pole) = −40 dB/dec.

ω = 1000 rad/sec. to higher frequency;

Slope = −40 + (−20) = −60 dB/dec.

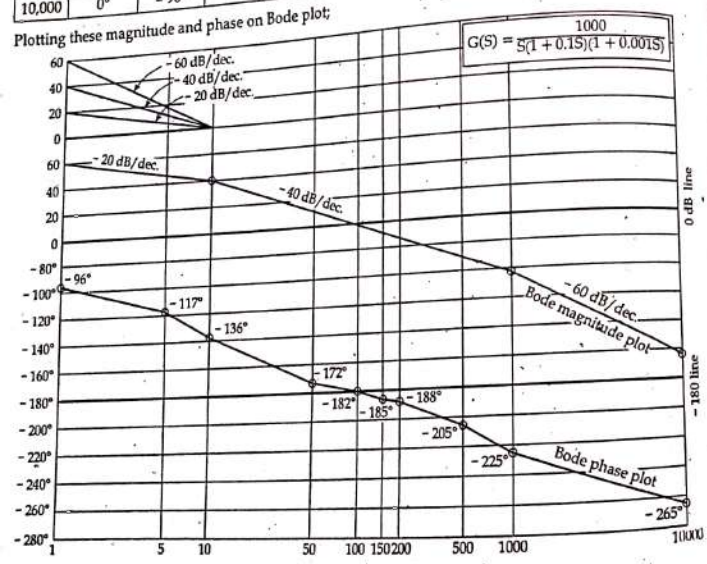
Scanned with CamScanner

Step 5 Phase plot

$\Phi_{Total} = -90° - \tan^{-1}(0.1\omega) - \tan^{-1}(0.001\omega)$

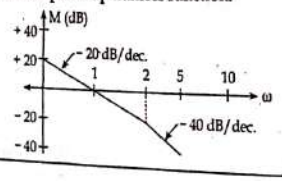
W	Arg 1000	Arg (jω)	$-\tan^{-1}(0.1\omega)$	$-\tan^{-1}(0.001\omega)$	Φ_{Total}
1	0°	- 90°	- 5.71°	- 0.050°	-95.76°
5	0°	- 90°	- 26.56°	- 0.286°	-116.84°
10	0°	- 90°	- 45°	- 0.572°	-135.57°
50	0°	- 90°	- 78.69°	- 2.86°	-171.55°
100	0°	- 90°	- 84.28°	- 5.71°	-180°
150	0°	- 90°	- 86.18°	- 8.53°	-184.71°
200	0°	- 90°	- 87.13°	- 11.30°	-188.43°
500	0°	- 90°	- 88.85°	- 26.56°	-205.41°
1000	0°	- 90°	- 89.43°	- 45°	-224.43°
10,000	0°	- 90°	- 89.94°	- 84.283°	-264.23°

Plotting these magnitude and phase on Bode plot;



EXAMPLE 6.2

From the given Bode plot, determine open loop transfer function. **[2068 Chaitra]**



Solution:

Step 1

The above diagram is Bode magnitude plot.

Step 2

Starting magnitude = 20 dB

We know that;

$$M = 20\log_{10} K - 20N\log_{10}\omega$$

and, ω = Starting frequency = 0.1 rad/sec. (Nearest decade scale)

so, $20 = 20\log_{10}(K) - 20N\log_{10}(0.1)$

Since, the initial slope is −20 dB/dec., therefore, the system is type-one system.

so, $N = 1$

i.e., $20 = 20\log_{10}(K) - 20\log_{10}(0.1)$

or, $20 = 20\log_{10}(K) + 20$

or, $20\log_{10}(K) = 0$

or, $\log_{10}(K) = 0$

$\therefore K = 1$

Step 3

The slope changed at $\omega = 2$ rad/sec.;

$\therefore$ Corner frequency $(\omega_1) = 2$ rad/sec.

$\therefore \omega = \frac{1}{T_1} = 2$ rad/sec.

Since, the slope is −40 dB/dec.; so, this is due to contribution of single pole.

so, Pole $= (1 + ST_1) = (1 + 0.5S)$

Step 4

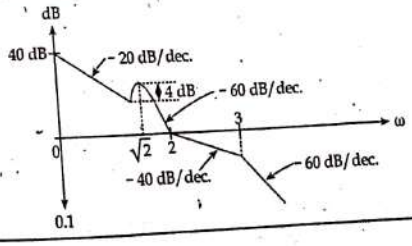
The final transfer function is;

$$G(S) = \frac{K}{S(1 + ST_1)}$$

$$\therefore \boxed{G(S) = \frac{1}{S(1 + 0.5S)}}$$

EXAMPLE 6.3

An engineer is called in to consult on a control system in a piece of equipment in the field. No one can find the design report or test results from the original design of control system. The engineer therefore decided to take a frequency response of the system. The resulting asymptotic frequency response is obtained as below. Determine the transfer function. **[2070 Chaitra]**



Solution:

Scanned with CamScanner

Step 1

The above diagram is Bode magnitude plot.

Step 2

Since, the initial slope is −20 dB/dec.; so, the system is type-one system.

Providing N = 1;

Starting frequency (ω) = 0.1 rad/sec.

Starting magnitude (dB) = $20\log_{10}K - 20N\log_{10}\omega$

or, $40 = 20\log_{10}K - 20\times1\times\log_{10}(0.1)$

or, $40 = 20\log_{10}K + 20$

or, $1 = \log_{10}K$

or, $\log_{10}(10) = \log_{10}(K)$

$\therefore$ $\boxed{K = 10}$

Step 3

At $\omega = \sqrt{2}$ rad/sec.; there is a second order system because the slope changes form −20 dB/dec. to −60 dB/dec. and also a peak of 4 dB is given.

$\therefore$ $\omega = \sqrt{2}$ rad/sec.

so, there is $\left[1-\left(\frac{\omega}{\omega_n}\right)^2 + j\cdot2\zeta\left(\frac{\omega}{\omega_n}\right)\right]$ in pole.

Now, taking magnitude at $\omega = \omega_n = \sqrt{2}$ rad/sec.

$$= \sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2} = \sqrt{\left[1-\left(\frac{\sqrt{2}}{\sqrt{2}}\right)^2\right]^2 + 4\zeta^2\left(\frac{\sqrt{2}}{\sqrt{2}}\right)^2} = 2\zeta \text{ in pole}$$

Now, magnitude of this factor is;

$$20\log_{10}\left(\frac{1}{2\zeta}\right) = 4 \text{ dB}$$

or, $\log_{10}\left(\frac{1}{2\zeta}\right) = \frac{1}{5}$

or, $\frac{1}{2\zeta} = (10)^{0.2}$

or, $\zeta = \frac{1}{2\times1.58} = \frac{1}{3.17}$

$\therefore$ $\boxed{\zeta = 0.316}$

Therefore, $\left[1+\left(\frac{S}{\omega_n}\right)^2 + 2\zeta\left(\frac{S}{\omega_n}\right)\right]$ in pole becomes;

$$\left[\left(1+\frac{S^2}{2}\right) + 2\times0.316\times\left(\frac{S}{\sqrt{2}}\right)\right] = (1+0.55^2+0.447S) \text{ in pole}$$

Step 4

At $\omega = 2$ rad/sec., the slope is −40 dB/dec., so there is zeros.

i.e., $\omega_a = \frac{1}{T_a} = 2$

$\therefore$ $T_a = \frac{1}{2} = 0.5$

This means $(1 + ST_a) = (1 + 0.5S)$ in zeros.

Step 5

At $\omega = 3$ rad/sec.; the slope is −60 dB/dec., so there is poles.

i.e., $\omega_1 = \frac{1}{T_1} = 3$

and, $T_1 = 0.33$

so, the factor $(1 + ST_1) = (1 + 0.33S)$ lies in the poles.

Step 6

The final transfer function is;

$$G(S) = \frac{K(1+ST_a)}{S(1+ST_1)\left[1+\left(\frac{S}{\omega_n}\right)^2 + 2\zeta\left(\frac{S}{\omega_n}\right)\right]}$$

$\therefore$ $\boxed{G(S) = \frac{10(1+0.5S)}{S(1+0.33S)\left(1+0.447S+0.5S^2\right)}}$; which is the required transfer function.

EXAMPLE 6.4

Sketch the Bode plot of $G(S) = \frac{512(S+3)}{S(S^2+16S+256)}$

Solution:

Step 1

$$G(S) = \frac{512(S+3)}{S(S^2+16S+256)} = \frac{6\left(1+\frac{1}{3}S\right)}{S\left[1+\frac{S}{16}+\left(\frac{S}{16}\right)^2\right]}$$

Step 2

Put, $S = j\omega$;

$$G(j\omega) = \frac{6\left(1+j\cdot\frac{\omega}{3}\right)}{j\omega\left[1-\left(\frac{\omega}{16}\right)^2\right] + j\cdot\left(\frac{\omega}{16}\right)}$$

i.e., type-one system with K = 6 and ω_n = 16 rad/sec.

Step 3 Calculation of corner frequencies

Corner frequency for zero $(\omega_1) = \frac{1}{\left(\frac{1}{3}\right)} = 3$ rad/sec.

Corner frequency for pole $(\omega_2) = \omega_n = 16$ rad/sec.

Now,

Starting frequency (ω) = $\frac{1}{10}\times$ Smallest corner frequency

$= \frac{1}{10}\times3$

$= 0.3 \approx 0.1$ rad/sec. (approximating to the nearest decade scale)

Step 4

Starting magnitude (dB) = $20\log_{10}K - 20N\log_{10}\omega$

N = Type of system = 1

K = 6

and, $\omega = 0.1$

i) Starting magnitude = $20\log_{10}(6) - 20\log_{10}(0.1) = 15.563 + 20 = 35.56$ dB

ii) From $\omega = 0.1$ rad/sec. to $\omega = 3$ rad/sec.;

Slope = −20 dB/sec.

From $\omega = 3$ rad/sec. to $\omega = 16$ rad/sec.;

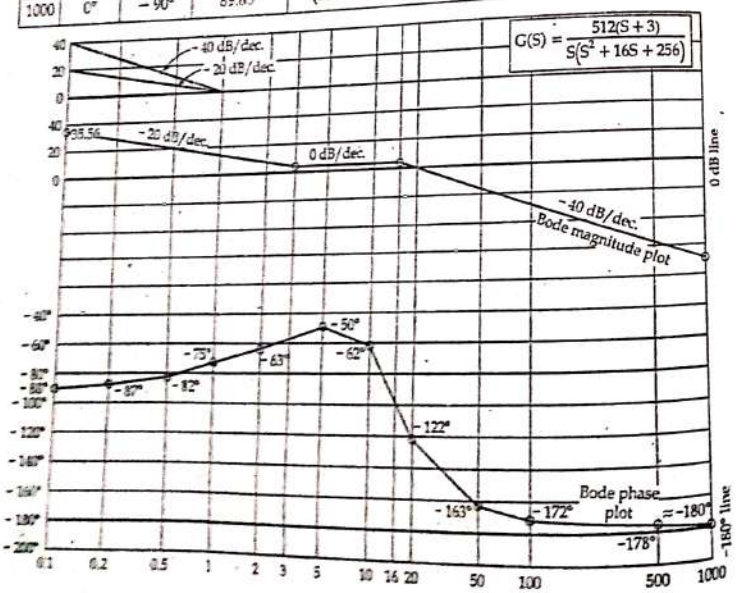
Scanned with CamScanner

Slope = −20 + 20 = 0 dB/sec.
From ω = 16 rad/sec. to ω = higher;
Slope = 0 + (−40) = −40 dB/sec.

Step 5 Phase plot

$$\angle G(j\omega) = -90° + \tan^{-1}\left(\frac{\omega}{3}\right) - \tan^{-1}\left(\frac{16\omega}{256-\omega^2}\right)$$

ω	Arg (6)	Arg (jω)	$+\tan^{-1}\left(\frac{\omega}{3}\right)$	$-\tan^{-1}\left(\frac{16\omega}{256-\omega^2}\right)$	Φ_{Total}
0.1	0°	−90°	1.9°	−0.358°	−88.45°
0.2	0°	−90°	3.81°	−0.716°	−86.90°
0.5	0°	−90°	9.46°	−1.79°	−82.33°
1	0°	−90°	18.43°	−3.59°	−75.16°
2	0°	−90°	33.7°	−7.23°	−63.53°
5	0°	−90°	59.07°	−19.10°	−50.67°
10	0°	−90°	73.30°	−45.72°	−62.42°
20	0°	−90°	81.46°	−(180° − 65°) = −114.2	−122.74°
50	0°	−90°	86.56°	−(180° − 19.63°) = −160.37°	−163.81°
100	0°	−90°	83.28°	−(180° − 9.33°) = −170.67°	−172.39°
500	0°	−90°	89.65°	−(180° − 1.84°) = −178.16°	−178.50°
1000	0°	−90°	89.85°	−(180° − 0.92°) = −179.08°	−179.25°



EXAMPLE 6.5

Draw Bode plot for the system with transfer function. Determine gain margin, phase margin, and comment on stability of the system according to your plot.

$$G(S) = \frac{4S+40}{(S^2+4S+25)(S^2+50S)}$$

[2072 Chaitra]

Solution:

Step 1

$$G(S) = \frac{40\left(1+\frac{4S}{40}\right)}{25\left(1+\frac{4S}{25}+\frac{S^2}{25}\right)\times S\times 50\left(1+\frac{S}{50}\right)} = \frac{40(1+0.1S)}{1250S(1+0.02S)\left[1+\left(\frac{4}{25}\right)S+\left(\frac{S}{5}\right)^2\right]}$$

$$= \frac{0.032(1+0.1S)}{S(1+0.02S)\left[1+\left(\frac{4}{25}\right)S+\left(\frac{S}{5}\right)^2\right]}$$

Step 2

Put, S = jω; then,

$$G(j\omega) = \frac{0.032(1+j0.1\omega)}{j\omega(1+j0.02\omega)\left[1-\left(\frac{\omega}{5}\right)^2+j\left(\frac{4}{25}\right)\omega\right]}$$

Hence, the system is first order (N = 1) with K = 0.032 and ω_n = 5 rad/sec..

Step 3 Calculate of corner frequency

Corner frequency for zeros;

$$\omega_1 = \frac{1}{T_1} = \frac{1}{0.1} = 10 \text{ rad/sec.}$$

Corner frequency for poles;

$$\omega_2 = \frac{1}{T_2} = \frac{1}{0.02} = 50 \text{ rad/sec.}$$

Comparing with $\left[1-\left(\frac{\omega}{\omega_n}\right)^2+j\cdot 2\zeta\left(\frac{\omega}{\omega_n}\right)\right]$; we have,

The second degree corner frequency for pole is;

$$\omega_3 = \omega_n = 5 \text{ rad/sec. (smallest)}$$

Step 4

i) Starting frequency (ω) = $\frac{1}{10}\times$ Smallest corner frequency = $\frac{1}{10}\times 5$
= 0.5 ≈ 0.1 rad/sec. (To make graph easy and accurate)

ii) Starting magnitude = $20\log_{10}K - 20N\log_{10}\omega = 20\log_{10}(0.032) - 20\log_{10}(0.1)$
= −29.9 + 20 = −9.9 ≈ −10 dB

Step 5

Form ω = 0.1 rad/sec. to ω = 5 rad/sec.;
Slope = −20 dB/sec.
Form ω = 5 rad/sec. to ω = 10 rad/sec.;
Slope = −20 + (−40) = −60 dB/sec.
Form ω = 10 rad/sec. to ω = 50 rad/sec.;
Slope = −60 + (20) = −40 dB/sec.
Form ω = 50 rad/sec. to ω = higher;
Slope = −40 + (−20) = −60 dB/sec.

Scanned with CamScanner

Step 6 Phase plot

$$\angle G(j\omega) = -90° - \tan^{-1}(0.02\omega) + \tan^{-1}(0.1\omega) - \tan^{-1}\left(\frac{4\omega}{25-\omega^2}\right)$$

ω	Arg (0.032)	Arg (jω)	$+\tan^{-1}(0.1\omega)$	$-\tan^{-1}(0.02\omega)$	$-\tan^{-1}\left(\frac{4\omega}{25-\omega^2}\right)$	Φ_{Total}
0.1	0°	− 90°	0.573°	− 0.1146°	− 0.917°	− 90.45°
0.2	0°	− 90°	1.145°	− 0.229°	− 1.835°	− 90.919°
0.5	0°	− 90°	2.862°	− 0.573°	− 4.62°	− 92.33°
1	0°	− 90°	5.71°	− 1.1458°	− 9.462°	− 94.89°
2	0°	− 90°	11.301°	− 2.29°	− 20.85°	− 101.84°
5	0°	− 90°	26.565°	− 5.71°	− 90°	− 159.145°
10	0°	− 90°	45°	− 11.310°	− (180–28°) = − 152°	− 208°
20	0°	− 90°	63.435°	− 21.80°	− (180–12°) = − 168°	− 216°
50	0°	− 90°	78.69°	− 45°	− (180–4.62) = − 175.38	− 231.7°
100	0°	− 90°	84.289°	− 63.435°	− (180–2.3) = − 177.7	− 247°
500	0°	− 90°	88.854°	− 84.29°	− (180–0.45) = − 179.55	− 265°
1000	0°	− 90°	89.427°	− 87.1376°	− (180–0.29) = − 179.77	− 268°

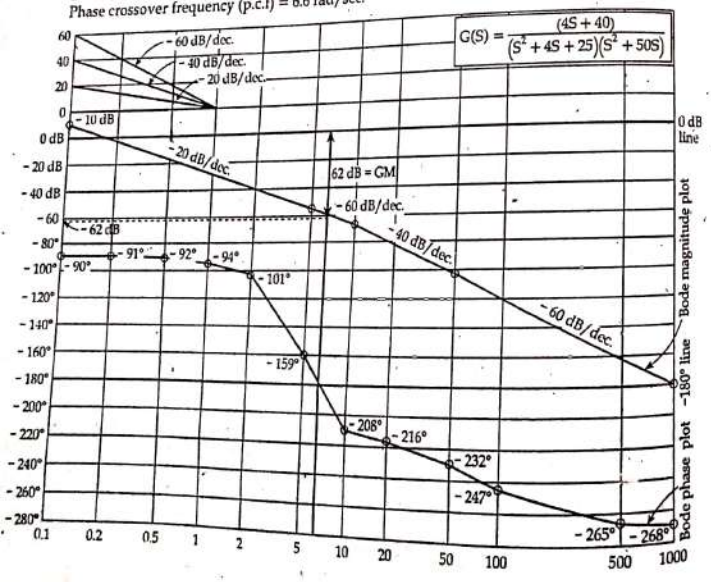
From semi-log paper (Bode plot); we have,

Gain margin (GM) = 0 − (−62) = 62 dB

Phase margin (PM) = We can't measure because no g.c.f.

g.c.f (gain cross over frequency) = infinite because magnitude plot doesn't intersect 0 dB line

Phase crossover frequency (p.c.f) = 6.6 rad/sec.



6.3.5 Minimum Phase System and Non–Minimum Phase Systems

The transfer function having poles and zeros in the left half of S-plane is called minimum phase transfer functions. Systems with minimum phase transfer functions are called minimum phase systems. For *example;* $G_1(S) = \frac{(1+ST_1)}{(1+ST_2)}$

The transfer function having poles and/or zeros in the right half of S-plane is called non-minimum phase transfer functions. Systems with non-minimum phase transfer functions are called non-minimum phase systems. *For example;* $G_2(S) = \frac{(1-ST_1)}{(1-ST_2)}$

6.4 EFFECT OF GAIN AND TIME CONSTANT ON BODE PLOT

6.4.1 Effect of Gain (K)

Let, $G(S) = \frac{K}{(1+S)}$

I) On magnitude plot

$$|G(j\omega)| = \frac{|K|}{|1+j\omega|} = \frac{K}{\sqrt{1+\omega^2}}$$

$$\text{Magnitude} = 20\log_{10}|G(j\omega)| = 20\log_{10}K - 20\log_{10}\sqrt{1+\omega^2}$$

- When K increases, then $20\log_{10}K$ also increases; so, magnitude plot shifts upward.
- When K decreases, then $20\log_{10}K$ also decreases; so, magnitude plot shifts downward.

Hence, increasing gain (K) increases the magnitude and magnitude plot shifts upwards.

II) On phase plot

Angle of gain K is 0°.

$\angle K = 0°$

So, gain has no effect on phase plot.

6.4.2 Effect of Time Constant

Let, $G(S) = e^{-TS}$

Putting $S = j\omega$;

$$G(j\omega) = e^{-j\omega T} = \cos\omega T - j\sin\omega T$$

$$|G(j\omega)| = \sqrt{\cos^2\omega T + \sin^2\omega T} = 1$$

I) On magnitude plot

$$\text{Magnitude (M)} = 20\log_{10}|G(j\omega)| = 20\log_{10}(1) = 0\ \text{dB}$$

Time constant hasn't any effect on magnitude plot.

II) On phase plot

$$\angle G(j\omega) = \tan^{-1}\left(-\frac{\sin\omega T}{\cos\omega T}\right) = \tan^{-1}(-\tan\omega T)$$

$$= -\tan^{-1}(\tan\omega T)$$

$$= -\omega T = -\omega T^{C}\ \text{(radian)}$$

$$= -57.3\ \omega T\ \text{(degree)}$$

Hence, phase plot lags by ωT (radian) *i.e.*, time delay can be introduced without affecting the magnitude.

Scanned with CamScanner

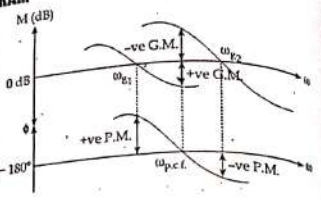
6.5 PHASE MARGIN, GAIN MARGIN, PHASE CROSSOVER FREQUENCY, GAIN CROSSOVER FREQUENCY AND STABILITY FROM BODE DIAGRAM

i) Gain crossover frequency (g.c.f)

The frequency at which magnitude plot crosses the 0 dB line is called gain crossover frequency. It is denoted by ω_{g_1} and ω_{g_2} in the figure.

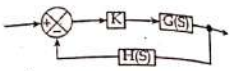
ii) Phase crossover frequency (p.c.f)

The frequency at which phase plot crosses the −180° line is called phase crossover frequency. It is denoted by ω_{pcf} in the figure.



iii) Gain margin (GM)

Gain margin is defined as the margin in gain allowable by which gain can be increased till the system reaches on the point of instability.



Let, for a system;

$$\text{T.F.} = \frac{K\,G(S)}{1 + K\,G(S)H(S)}$$

The characteristics equation is;

$$1 + K\,G(S)H(S) = 0$$

or, $K\,G(S)H(S) = -1$

or, $|K\,G(S)H(S)| = 1$

$\therefore$ $\boxed{K = \frac{1}{|G(S)H(S)|}}$, which is the gain margin (G.M.).

Now, gain margin can written in decibel;

so, $\text{Gain} = 20\log_{10} K = 20\log_{10}\left(\frac{1}{|G(S)H(S)|}\right) = -20\log_{10}|G(S)H(S)|$ in dB

Gain margin is always measured form 0 dB line.

$$\text{G.M.} = -20\log_{10}|G(j\omega)|_{\omega=\omega_{p.c.f.}} \text{ decibel}$$

The magnitude plot gives gain margin, which describes the magnitude plot will be stable or unstable.

i) G.M. = +ve → Stable (if g.c.f < p.c.f)

ii) G.M. = −ve → Unstable (if g.c.f > p.c.f)

iii) G.M. = 0 → Marginally stable (g.c.f = p.c.f)

In terms of gain (K)

$$\text{G.M.} = 20\log_{10} K = -20\log_{10}|G(S)H(S)|_{\omega=p.c.f.}$$

i) G.M. = +ve if K > 1 and → Stable

ii) G.M. = −ve if K < 1 and → Unstable

iii) G.M. = 0 if K = 1 and → Marginally stable

v) Phase margin (PM)

Phase margin is the amount of additional phase lag which can be introduced in the system till it reaches the point of instability.

Phase margin (P.M.) can be obtained from phase plot and measured form − 180° line. The unit of phase margin is degree. Mathematically;

$$\text{P.M.} = \angle G(j\omega)H(j\omega)|_{\text{at } \omega=g.c.f.} - (-180°) = 180° + \phi$$

where, $\phi = \angle G(j\omega)H(j\omega)$ at gain cross over frequency.

The phase plot will be stable or unstable. It can be described through phase margin (P.M).

i) If P.M. = +ve → System stable (only phase plot)

ii) If P.M. = −ve → System unstable (only phase plot)

iii) If P.M. = 0 → Marginally stable

NOTE Effect of temperature.

The transfer function (T.F.) changes with change of temperature and pressure. The gain of the system is also affected by the supply voltage, Supply frequency, and air pressures.

If the gain is high, the G.M. is low and for step response the settling time is large.

If the gain is low, gain margin and phase margin will be high and setting time will be more. *i.e.*, the system response will be sluggish and also the rise time and steady state error will be high.

Stability Analysis from Bode Plot

If,

Gain margin	Phase margin	Stability
+ve	+ve	Bode plot and System will be stable
+ve	−ve	System will be inherently stable
−ve	+ve	
−ve	−ve	System will be unstable
0	0	Marginally stable

6.6 POLAR PLOT AND STABILITY

6.6.1 Polar Plot

Polar plot of sinusoidal T.F. $G(j\omega)$ is a plot of magnitude of $G(j\omega)$ verses the phase angles of $G(j\omega)$ on polar coordinates. *i.e.*, plot in (r, θ) format as ω is varied from zero to infinity.

i.e., polar plot is the locus of vector $|G(j\omega)|\angle G(j\omega)$ as ω is varied from 0 to ∞.

$\angle G(j\omega)$ is positive for counterclockwise and negative for clockwise direction. All the angles are measured from positive real axis.

Advantages

It shows the frequency response of a system over the entire frequency range in single plot.

Disadvantages

It doesn't clearly indicate the contribution of each individual factor of open loop transfer function.

6.6.2 Procedure to Sketch Polar Plot

i) Obtain transfer function (T.F.) G(S) of the system

ii) Put, $S = j\omega$ to obtain $G(j\omega)$.

iii) Calculate magnitude of $G(j\omega)$ at $\omega = 0$ and $\omega = \infty$

$$\lim_{\omega \to 0} |G(j\omega)|$$

and, $$\lim_{\omega \to \infty} |G(j\omega)|$$

iv) Calculate phase angle of $G(j\omega)$ at $\omega = 0$ and $\omega = \infty$

$$\lim_{\omega \to 0} \angle G(j\omega)$$

Scanned with CamScanner

and, $\lim_{\omega \to \infty} \angle G(j\omega)$

v) Rationalize $G(j\omega)$ and separate real and imaginary part.

vi) Equate imaginary part to zero $[\text{Im}\{G(j\omega)\} = 0]$ to obtain frequencies at which plot intersects the real axis and calculate value of $G(j\omega)$ at the point of intersection by substituting the determined value of frequency in the expression of $G(j\omega)$.

vii) Equate real part with zero. $[\text{Re}\{G(j\omega)\} = 0]$ to obtain frequencies at which plot intersects the imaginary axis and calculate value of $G(j\omega)$ at the point of intersection by substituting the determined value of frequency in the rationalized expression of $G(j\omega)$.

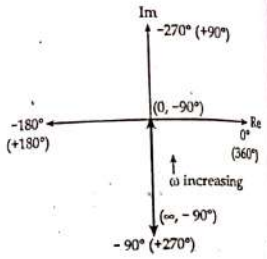
viii) Sketch polar plot with the help of above information.

6.6.3 Polar Plots of Some Standard Type Functions

i) $G(S) = \frac{1}{S}$

Put, $S = j\omega$

$$G(j\omega) = \frac{1}{j\omega} = -\frac{(-1)}{j\omega} = \frac{-j^2}{j\omega} = \frac{1}{\omega} \times (-j) = \left(\frac{1}{\omega}\right) \angle -90°$$



At $\omega = 0$;

$$|G(j\omega)| = \frac{1}{0} = \infty$$

$$\angle G(j\omega) = -90°$$

At $\omega = \infty$;

$$|G(j\omega)| = \frac{1}{\infty} = 0$$

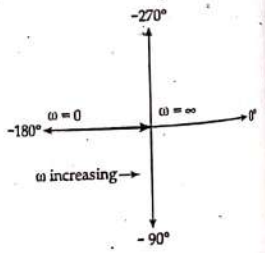
$$\angle G(j\omega) = -90°$$

Starts from $(\infty, -90°)$ at $\omega = 0$ and ends to $(0, -90°)$ at $\omega = \infty$.

ii) $G(S) = \frac{1}{S^2}$

Put, $S = j\omega$

$$G(j\omega) = \frac{1}{(j\omega)^2} = \frac{1}{-\omega^2} = \frac{1}{\omega^2} \angle -180°$$



At $\omega = 0$;

$$|G(j\omega)| = \frac{1}{0} = \infty$$

$$\angle G(j\omega) = -180°$$

At $\omega = \infty$;

$$|G(j\omega)| = \frac{1}{\infty} = 0$$

$$\angle G(j\omega) = -180°$$

Starts from $(\infty, -180°)$ at $\omega = 0$ and ends to $(0, -180°)$ at $\omega = \infty$.

iii) $G(S) = S$

Put, $S = j\omega$

$$G(j\omega) = j\omega = \omega j = \omega \angle 270°$$

At $\omega = 0$;

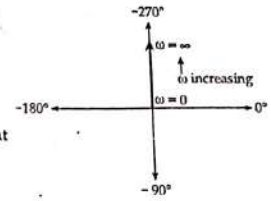
$$|G(j\omega)| = 0$$

$$\angle G(j\omega) = -270°$$

At $\omega = \infty$;

$$|G(j\omega)| = \infty$$

$$\angle G(j\omega) = -270°$$

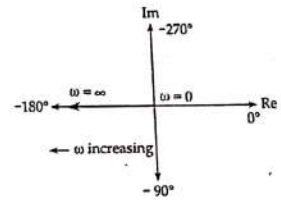


Starts from $(0, -270°)$ at $\omega = 0$ and ends to $(\infty, -270°)$ at $\omega = \infty$.

iv) $G(S) = S^2$

Put, $S = j\omega$

$$G(j\omega) = (j\omega)^2 = -\omega^2 = \omega^2 \angle -180°$$



At $\omega = 0$;

$$|G(j\omega)| = \omega^2 = 0$$

$$\angle G(j\omega) = -180°$$

At $\omega = \infty$;

$$|G(j\omega)| = \omega^2 = \infty$$

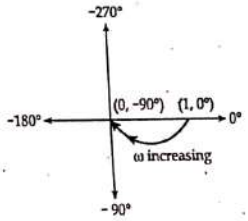
$$\angle G(j\omega) = -180°$$

Starts from $(0, -180°)$ at $\omega = 0$ and ends to $(\infty, -180°)$ at $\omega = \infty$.

v) $G(S) = \frac{1}{S+1}$

Put, $S = j\omega$

$$G(j\omega) = \frac{1}{j\omega + 1} = \frac{1}{\sqrt{1+\omega^2}} = \angle -\tan^{-1}(\omega)$$



At $\omega = 0$;

$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2}} = 1$$

$$\angle G(j\omega) = -\tan^{-1}(0) = 0°$$

At $\omega = \infty$;

$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2}} = 0$$

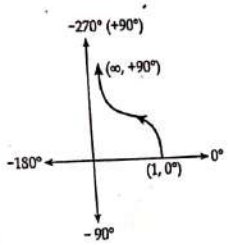
$$\angle G(j\omega) = -\tan^{-1}(\infty) = -90°$$

Starts from $(1, 0°)$ at $\omega = 0$ and ends to $(0, -90°)$ at $\omega = \infty$.

vi) $G(S) = 1 + S$

Put, $S = j\omega$

$$G(j\omega) = 1 + j\omega = \sqrt{1+\omega^2} \angle \tan^{-1}(\omega)$$



At $\omega = 0$;

$$|G(j\omega)| = \sqrt{1+\omega^2} = 1$$

$$\angle G(j\omega) = \tan^{-1}(0) = 0°$$

At $\omega = \infty$;

$$|G(j\omega)| = \sqrt{1+\omega^2} = \infty$$

$$\angle G(j\omega) = \tan^{-1}(\infty) = +90°$$

Starts from $(1, 0°)$ at $\omega = 0$ and ends to $(\infty, +90°)$ at $\omega = \infty$.

Scanned with CamScanner

6.6.4 Polar Plot of Type Zero System

Let, $G(S) = \frac{K}{(1+ST_1)(1+ST_2)}$

Step 1

Put, $S = j\omega$

$$G(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)} = \frac{K}{\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}} \angle -\tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

Step 2

$$\lim_{\omega \to 0} |G(j\omega)| = K$$

$$\lim_{\omega \to 0} \angle G(j\omega) = -\tan^{-1}(0) - \tan^{-1}(0) = 0°$$

Step 3

$$\lim_{\omega \to \infty} |G(j\omega)| = 0$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = -\tan^{-1}(\infty) - \tan^{-1}(\infty) = -2\tan^{-1}(\infty) = -180°$$

Step 4 Rationalization

$$G(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)} \times \frac{1-j\omega T_1}{1-j\omega T_1} \times \frac{1-j\omega T_2}{1-j\omega T_2} = \frac{K(1-j\omega T_1)(1-j\omega T_2)}{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}$$

$$= \frac{K(1-\omega^2T_1T_2)}{(1+\omega^2T_1^2)(1+\omega^2T_2^2)} - j\frac{K\omega(T_1+T_2)}{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}$$

Step 5

Equating $Re\{G(j\omega)\} = 0$;

or, $$\frac{K(1-\omega^2T_1T_2)}{(1+\omega^2T_1^2)(1+\omega^2T_2^2)} = 0$$

Either, $\omega = \pm\infty$ or, $(1-\omega^2T_1T_2) = 0$

or, $\omega^2 = \frac{1}{T_1T_2}$

$\therefore \quad \omega = +\frac{1}{\sqrt{T_1T_2}}$

This is the frequency at which polar plot intersects in the imaginary axis. We only consider the positive value of ω since polar plot is the locus of point ranging from $\omega = 0$ to $\omega = +\infty$.

Now, value can be found by;

$$G(j\omega) = 0 - j\frac{K\omega(T_1+T_2)}{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}$$

Put, $\omega = \frac{1}{\sqrt{T_1T_2}}$

$$G(j\omega) = -\frac{jK\left(\frac{1}{\sqrt{T_1T_2}}\right)(T_1+T_2)}{\left(1+\frac{T_1^2}{T_1T_2}\right)\left(1+\frac{T_2^2}{T_1T_2}\right)} = -\frac{jK(T_1+T_2)}{\sqrt{T_1T_2}\left(1+\frac{T_1}{T_2}\right)\left(1+\frac{T_2}{T_1}\right)} = -\frac{jK\sqrt{T_1T_2}(T_1+T_2)}{(T_1+T_2)^2} = \frac{K\sqrt{T_1T_2}}{T_1+T_2} \angle -90°$$

Step 6

Equating $Im\{G(j\omega)\} = 0$;

$$-\frac{jK\omega(T_1+T_2)}{(1+\omega^2T_1^2)(1+\omega^2T_2^2)} = 0$$

Either, $\omega = \pm\infty$ or $\omega = 0$

Step 7 Sketch of polar plot

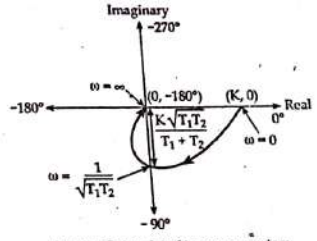


Figure: Polar plot of type zero system

EXAMPLE 6.6

Sketch polar plot of $G(S) = \frac{20}{(S+4)(S+5)}$

Solution:

Step 1

Put, $S = j\omega$

$$G(j\omega) = \frac{20}{(j\omega+4)(j\omega+5)} = \frac{20}{\sqrt{\omega^2+16}\sqrt{\omega^2+25}} \angle -\tan^{-1}\left(\frac{\omega}{4}\right) - \tan^{-1}\left(\frac{\omega}{5}\right)$$

Step 2

$$\lim_{\omega \to 0} |G(j\omega)| = \frac{20}{4\times 5} = 1$$

$$\lim_{\omega \to 0} \angle G(j\omega) = -\tan^{-1}(0) - \tan^{-1}(0) = 0°$$

Step 3

$$\lim_{\omega \to \infty} |G(j\omega)| = \frac{20}{\infty} = 0$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = -\tan^{-1}(\infty) - \tan^{-1}(\infty) = -2\tan^{-1}(\infty) = -180°$$

Step 4 Rationalization

$$G(j\omega) = \frac{20}{(j\omega+4)(j\omega+5)} \times \frac{4-j\omega}{4-j\omega} \times \frac{5-j\omega}{5-j\omega} = \frac{20(4-j\omega)(5-j\omega)}{(16+\omega^2)(25+\omega^2)}$$

$$= \frac{20(20-\omega^2)}{(16+\omega^2)(25+\omega^2)} - j\frac{20\times 9\omega}{(16+\omega^2)(25+\omega^2)}$$

Step 5

Equating $Re\{G(j\omega)\} = 0$

Either, $$\frac{20(20-\omega^2)}{(16+\omega^2)(25+\omega^2)} = 0 \quad \text{or,} \quad \omega = \pm\infty$$

Scanned with CamScanner

or, $20 - \omega^2 = 0$

or, $\omega^2 = 20$

$\therefore \quad \omega = \sqrt{20} = 2\sqrt{5}$ rad/sec.

At this value of ω we can calculate the intercept on imaginary axis.

$$G(j\omega) = 0 - j\frac{20 \times 9\omega}{(16+\omega^2)(25+\omega^2)} = -j\frac{180 \times 2\sqrt{5}}{(16+20)(25+20)} = -j\left(\frac{360\sqrt{5}}{1620}\right) = 0.497(-j) = 0.497\angle -90°$$

This is the value of intercept on $(-j)$ axis.

Step 6

Equating $\text{Im}\{G(j\omega)\} = 0$

or, $-\frac{180\omega}{(16+\omega^2)(25+\omega^2)} = 0$

Either, $\omega = 0$ or $\omega = \pm\infty$

No, distinct value of ω.

Step 7

The polar plot becomes;

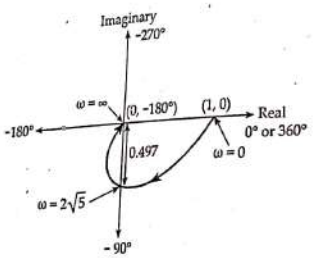


Figure: Polar plot

6.6.5 Polar Plot of Type I System

Let, $G(S) = \frac{K}{S(1+ST_1)(1+ST_2)} = \frac{K}{\omega\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}}\angle -90° - \tan^{-1}(\omega T_1)° - \tan^{-1}(\omega T_2)°$

Step 1

Put, $S = j\omega$

$$G(j\omega) = \frac{K}{S(1+j\omega T_1)(1+j\omega T_2)}$$

Step 2

$$\lim_{\omega \to 0} |G(j\omega)| = \frac{K}{0} = \infty$$

$$\lim_{\omega \to 0} \angle G(j\omega) = -90°$$

Step 3

$$\lim_{\omega \to \infty} |G(j\omega)| = \frac{K}{\infty} = 0$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = -90° - 90° - 90° = -270°$$

Step 4 Rationalization

$$G(j\omega) = \frac{K}{j\omega(1+j\omega T_1)(1+j\omega T_2)} \times \frac{1-j\omega T_1}{1-j\omega T_1} \times \frac{1-j\omega T_2}{1-j\omega T_2} = \frac{K(1-j\omega T_1)(1-j\omega T_1)}{j\omega\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)}$$

$$= \frac{K(1-\omega^2T_1T_2) - jK\omega(T_1+T_2)}{j\omega\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)} = -\frac{K(T_1+T_2)}{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)} - j\frac{K(1-\omega^2T_1T_2)}{\omega\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)}$$

Step 5

Equating $\text{Re}\{G(j\omega)\} = 0$

or, $-\frac{K(T_1+T_2)}{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)} = 0;$

which gives $\omega = \pm\infty$ and no distinct value.

Step 6

Equating $\text{Im}\{G(j\omega)\} = 0$

$$-\frac{jK(1-\omega^2T_1T_2)}{\omega\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)}$$

Either, $\omega = \pm\infty$ or, $1 - \omega^2T_1T_2 = 0$

or, $\omega^2T_1T_2 = 1$

$\therefore \quad \omega = \frac{1}{\sqrt{T_1T_2}}$

This is the required frequency at which polar plot crosses the real axis and the intercept at real axis can be found by;

$$G(j\omega) = -\frac{K(T_1+T_2)}{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)} - 0$$

Substituting $\omega^2 = \frac{1}{T_1T_2}$; we get,

$$G(j\omega) = -\frac{K(T_1+T_2)}{\left(1+\frac{T_1}{T_2}\right)\left(1+\frac{T_2}{T_1}\right)} = -\frac{KT_1T_2(T_1+T_2)}{(T_1+T_2)^2} = -\frac{KT_1T_2}{(T_1+T_2)} = \frac{KT_1T_2}{(T_1+T_2)}\angle -180°$$

Therefore, polar plot intersects the real axis on negative half.

Step 7

The polar plot now becomes;

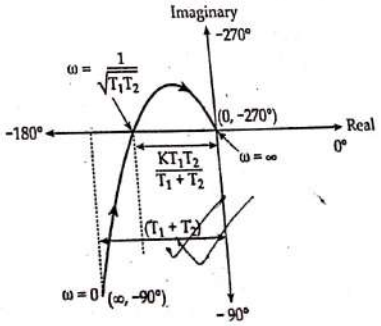


Figure: Polar plot of type–I system

Scanned with CamScanner

6.6.6 Polar Plot of Type –II System

Let, $G(S) = \frac{K}{S^2(1+ST_1)}$

Step 1

Put, $S = j\omega$

$$G(j\omega) = \frac{K}{S^2(1+j\omega T_1)} = \frac{K}{\omega^2\sqrt{1+\omega^2T_1^2}} \angle -180° - \tan^{-1}(\omega T_1)$$

Step 2

$$\lim_{\omega \to 0} |G(j\omega)| = \infty$$

$$\lim_{\omega \to 0} \angle G(j\omega) = -180°$$

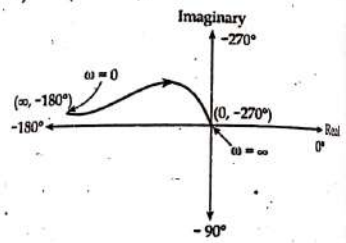
Step 3

$$\lim_{\omega \to \infty} |G(j\omega)| = 0$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = -180° - \tan^{-1}(\infty) = -180° - 90° = -270°$$

Step 4 Rationalization

$$G(j\omega) = -\frac{K}{\omega^2(1+j\omega T_1)} \times \frac{1-j\omega T_1}{1-j\omega T_1} = -\frac{K}{\omega^2\left(1+\omega^2T_1^2\right)} + j\frac{KT_1}{\omega\left(1+\omega^2T_1^2\right)}$$



Step 5

Equating $\mathrm{Re}\{G(j\omega)\} = 0$

$\omega = \pm\infty$ (no distinct value)

Step 6

Equating $\mathrm{Im}\{G(j\omega)\} = 0$

$\omega = \pm\infty$ (no distinct value)

Step 7

Hence, polar plot do not intersect any axis.

6.6.7 Effect of Additional Pole on Type –II System

Let, $G(S) = \frac{K}{S^2(1+ST_1)(1+ST_2)}$

Step 1

Put, $S = j\omega$

$$G(S) = -\frac{K}{\omega^2(1+j\omega T_1)(1+j\omega T_2)} = \frac{K}{\omega^2\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}} \angle -180° - \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

Step 2

$$\lim_{\omega \to 0} |G(j\omega)| = \infty$$

$$\lim_{\omega \to 0} \angle G(j\omega) = -180°$$

Step 3

$$\lim_{\omega \to \infty} |G(j\omega)| = 0$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = -360° = 0°$$

Step 4 Rationalization

$$G(j\omega) = -\frac{K}{\omega^2(1+j\omega T_1)(1+j\omega T_2)} \times \frac{1-j\omega T_1}{1-j\omega T_1} \times \frac{1-j\omega T_2}{1-j\omega T_2} = -\frac{K(1-j\omega T_1)(1-j\omega T_2)}{\omega^2\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)}$$

$$= -\frac{K(1-\omega^2T_1T_2)}{\omega^2\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)} + \frac{j\omega K(T_1+T_2)}{\omega^2\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)}$$

$$= \frac{K(\omega^2T_1T_2-1)}{\omega^2\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)} + \frac{jK(T_1+T_2)}{\omega\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)}$$

Step 5

Equating $\mathrm{Re}\{G(j\omega)\} = 0$

$$\frac{K(\omega^2T_1T_2-1)}{\omega^2\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)} = 0$$

Either, $\omega = \pm\infty$ or, $\omega^2T_1T_2 - 1 = 0$

$$\therefore \quad \omega = \frac{1}{\sqrt{T_1T_2}}$$

This is the value of ω where polar plot intersects the imaginary axis. The value of intercepts can be calculated by;

$$G(j\omega) = 0 + \frac{jK(T_1+T_2)}{\omega\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)} = \frac{jK(T_1+T_2)}{\frac{1}{\sqrt{T_1T_2}}\left(1+\frac{T_1^2}{T_1T_2}\right)\left(1+\frac{T_2^2}{T_1T_2}\right)} = \frac{jK\sqrt{T_1T_2}(T_1+T_2)}{\frac{(T_1+T_2)^2}{T_1T_2}}$$

$$= \frac{jK(T_1T_2)^{\frac{3}{2}}}{(T_1+T_2)} = \frac{jK(T_1T_2)^{\frac{3}{2}}}{(T_1+T_2)} \angle(-270°)$$

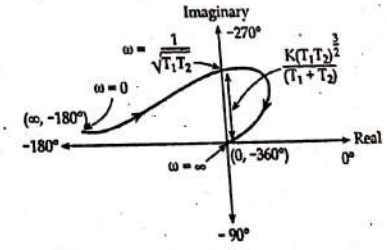
Step 6

Equating $\mathrm{Im}\{G(j\omega)\} = 0$

$\omega = \pm\infty$ (no distinct value)

Step 7

Polar plot now becomes;



NOTE

Hence addition of pole on type-II system, the polar plot intersects the axis. On every addition of poles; there will be more intersection in the axis.

Scanned with CamScanner

6.6.8 Polar Plot Involving Quadratic Factor

Let, $G(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$

Step 1

Put, $S = j\omega$

$$G(S) = \frac{\omega_n^2}{-\omega^2 + \omega_n^2 + 2\zeta\omega_n(j\omega)} = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + j(2\zeta\omega\omega_n)} = \frac{\omega_n^2}{\omega_n^2\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\omega\omega_n}{\omega_n^2}\right)\right]}$$

$$= \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_n}\right)} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}} \angle -\tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2 - \omega^2}\right)$$

Step 2

$$\lim_{\omega \to 0} |G(j\omega)| = 1$$

$$\lim_{\omega \to 0} \angle G(j\omega) = 0°$$

Step 3

$$\lim_{\omega \to \infty} |G(j\omega)| = 0$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = -180°$$

Step 4 Rationalization

$$G(j\omega) = \frac{1}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_n}\right)\right]} \times \frac{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 - j2\zeta\left(\frac{\omega}{\omega_n}\right)\right]}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 - j2\zeta\left(\frac{\omega}{\omega_n}\right)\right]} = \frac{\left[\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right] - j2\zeta\left(\frac{\omega}{\omega_n}\right)\right]}{\left[\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right]}$$

$$= \frac{1 - \left(\frac{\omega}{\omega_n}\right)^2}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 - 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2} - \frac{j2\zeta\left(\frac{\omega}{\omega_n}\right)}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}$$

Step 5

Equating $Re\{G(j\omega)\} = 0$

$$1 - \left(\frac{\omega}{\omega_n}\right)^2 = 0$$

or, $1 = \left(\frac{\omega}{\omega_n}\right)^2$

$\therefore$ $\boxed{\omega = \omega_n}$ which is the distinct point where polar plot intersects at imaginary axis.

and, the intercept can be calculated as;

$$G(j\omega) = 0 - \frac{j2\zeta\left(\frac{\omega}{\omega_n}\right)}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}$$

Substituting $\omega = \omega_n$, we get,

$$G(j\omega) = -\frac{j2\zeta}{4\zeta^2} = -\frac{j}{2\zeta} = \left(\frac{1}{2\zeta}\right)(-j) = \left(\frac{1}{2\zeta}\right)\angle(-90°)$$

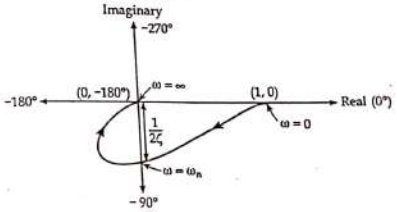
Step 6

Equating $Im\{G(j\omega)\} = 0$

$\omega = \pm\infty, \omega = 0$

No distinct point so polar do not intersects real axis.

Step 7 Sketch polar plot including above information



6.6.9 Stability Analysis from Polar Plot

i) Phase crossover frequency (p.c.f.)

The frequency that corresponds to the point of intersection of polar plot with negative real axis (*i.e.*, −180°) is called phase crossover frequency (p.c.f.).

ii) Gain crossover frequency (g.c.f.)

The frequency at which the magnitude of open loop sinusoidal transfer function *i.e.*, $G(j\omega)$ of a closed loop system is unity.

iii) Phase margin (PM)

The amount of additional phase lag which can be introduced in the system till it reaches the point of instability.

$$P.M. = 180° + \angle G(j\omega)\big|_{at\ \omega = g.c.f.}$$

$\therefore$ $P.M. = 180° + \phi$

For polar plot 1

$\phi < -180°$ (in magnitude)

and, $P.M. = 180° + \phi$

$\therefore$ $P.M. = +ve$

For polar plot 2

$\phi > -180°$

$P.M. = 180° + \phi$

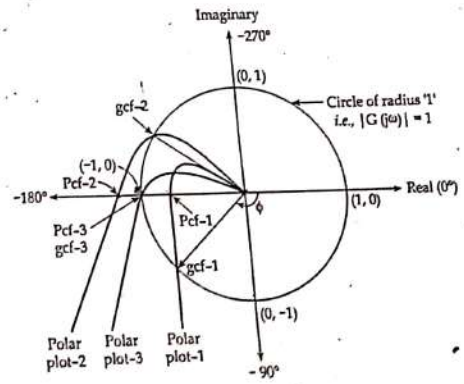
$\therefore$ $P.M. = -ve$

For polar plot 3

$\phi = -180°$

$P.M. = 180° + \phi$

$\therefore$ $P.M. = 0$



iv) Gain margin (G.M.)

Gain margin is defined as the margin in gain allowable by which gain can be increased till the system reaches on the point of instability.

We know that;

$$\text{Gain margin (G.M.)} = 20\log_{10} K = 20\log_{10}\left|\frac{1}{G(j\omega)}\right| = -20\log_{10}|G(j\omega)|_{p.c.f.} \text{ in dB}$$

Scanned with CamScanner

For polar plot 1

The p.c.f. lies inside the circle and $|G(j\omega)| < 1$

so, G.M. $= -20\log_{10}|G(j\omega)|_{p.c.f.}$

$\therefore$ G.M. $= +ve$

For polar plot 2

The p.c.f. lies outside the circle and $|G(j\omega)| > 1$.

so, G.M. $= -20\log_{10}|G(j\omega)|_{p.c.f.}$

$\therefore$ G.M. $= -ve$

For polar plot 3

The p.c.f. and g.c.f. lies on the circle of radius $|G(j\omega)| = 1$.

so, G.M. $= -20\log_{10}|G(j\omega)|_{p.c.f.} = 0$ dB

i) If plot doesn't crosses negative real axis then we can't get p.c.f. so G.M. is infinite.

i.e., $|G(j\omega)|_{p.c.f.} = 0$

so, Gain margin (G.M.) $= K = \frac{1}{|G(j\omega)|} = \infty$

ii) If plot intersects in negative real axis at a point between origin and $(-1, 0)$, then gain margin is greater than one.

Since, $0 < |G(j\omega)| < 1$; then,

$K = \frac{1}{|G(j\omega)|} = (> 1)$

iii) If plot intersects negative real axis away from $(-1, 0)$, gain margin is less than one.

$|G(j\omega)| > 1$

so, $K = \frac{1}{|G(j\omega)|} = (< 1)$

iv) If plot intersects at $(-1, 0)$; then, G.M. = Unity

$|G(j\omega)| = 1$

so, $K = \frac{1}{|G(j\omega)|} = 1$

v) Stability analysis

P.M.	G.M.	Stability
+ ve	+ ve	Stable
− ve	− ve	Unstable
0	0	Marginally stable
+ ve	− ve	Inherently stable
− ve	+ ve	Inherently stable
Any	∞	Inherently stable

EXAMPLE 6.7

Sketch approximate polar plot for a unity feedback system with feed forward transfer function $G(S) = \frac{10}{S(S+1)^2}$ and obtain: [2068 Baishakh]

i) P.M. ii) G.M. iii) P.c.f. iv) G.c.f. v) Stability

Solution:

Step 1

$G(S) = \frac{10}{S(S+1)^2}$

Put $S = j\omega$

$G(j\omega) = \frac{10}{j\omega(1+j\omega)^2} = \frac{10}{\omega\left(\sqrt{1+\omega^2}\right)^2}\angle -90° - 2\tan^{-1}\omega = \frac{10}{\omega(1+\omega^2)}\angle -90° - 2\tan^{-1}\omega$

Step 2

$\lim_{\omega \to 0}|G(j\omega)| = \infty$

$\lim_{\omega \to 0}\angle G(j\omega) = -90°$

Step 3

$\lim_{\omega \to \infty}|G(j\omega)| = 0$

$\lim_{\omega \to \infty}\angle G(j\omega) = -270°$

Step 4 Rationalization

$G(j\omega) = \frac{10}{j\omega(1+j\omega)^2} \times \frac{(1-j\omega)^2}{(1-j\omega)^2} = \frac{10(1-\omega^2-j2\omega)}{j\omega(1+\omega^2)^2} = \frac{10(1-\omega^2)}{j\omega(1+\omega^2)^2} - \frac{j20\omega}{j\omega(1+\omega^2)^2}$

$= -\frac{20}{(1+\omega^2)^2} - j\frac{10(1-\omega^2)}{\omega(1+\omega^2)^2}$

Step 5

Equating $\mathrm{Re}\{G(j\omega)\} = 0$

$\omega = \pm\infty$; no distinct values

Step 6

Equating $\mathrm{Im}\{G(j\omega)\} = 0$

$-j\frac{10(1-\omega^2)}{\omega(1+\omega^2)^2} = 0$

Either, $\omega = \pm\infty$ or, $1 - \omega^2 = 0$

$\therefore \omega = 1$ rad/sec.

This is the frequency at which polar plot intersects the real axis. The interest can be calculated by;

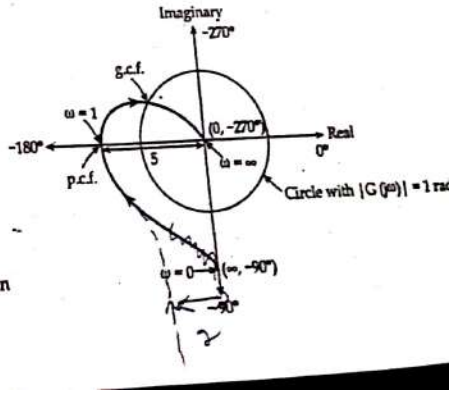
$G(j\omega) = -\frac{20}{(1+\omega^2)^2} - 0$

$= -\frac{20}{(1+1)^2}$

$= -\frac{20}{4}$

$= -5$

$= 5\angle -180°$



Step 7

The polar plot now becomes as shown aside;

Scanned with CamScanner

Step 8 Solution

i) Phase crossover frequency (P.c.f.) $= \omega = 1$ rad/sec

ii) Gain margin (G.M.) $= K = \frac{1}{|G(j\omega)|} = \frac{1}{5} = 0.2$

In dB,

$$G.M. = 20\log_{10}\left[\frac{1}{|G(j\omega)|}\right] = 20\log_{10}(0.2) = -13.97 \approx -14 \text{ dB}$$

iii) Now, at this frequency; gain cross over frequency (g.c.f.) can be calculated by:

$$|G(j\omega)| = 1$$

or, $\frac{10}{\omega(1+\omega^2)} = 1$

or, $10 = \omega + \omega^3$

or, $\omega^3 + \omega - 10 = 0$

Solving; we get,

$\omega_1 = 2$ rad/sec. (Valid)

$\omega_2 = -1 + 2j$

$\omega_3 = -1 - 2j$ } Complex invalid

$\therefore$ g.c.f. $= 2$ rad/sec.

iv) Phase margin (P.M.)

The g.c.f. gives the phase margin.

so, $\angle G(j\omega) = -90° - [\tan^{-1}(\omega)] \times 2 = -90° - 2 \times [\tan^{-1}(2)] = -216.87°$

and, Phase margin (P.M.) $= 180° + \angle G(j\omega) = 180° - 216.87° = -36.87°$

v) Since, P.M. = −ve, G.M. = −ve; so, the system is unstable.

EXAMPLE 6.8

$G(S) = \frac{K}{S(S+2)(S+10)}$

Determine value of K so that the system may be stable with:

a) G.M. = 6 dB and

b) P.M. = + 45°

Solution:

Step 1 Drawing polar plot

Put, $S = j\omega$; then,

$$G(j\omega) = \frac{K}{j\omega(j\omega+2)(j\omega+10)} = \frac{K}{\omega(\omega^2+4)(\omega^2+100)} \angle -90° - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{10}\right)$$

Step 2

$$\lim_{\omega \to 0} |G(j\omega)| = \infty$$

$$\lim_{\omega \to 0} \angle G(j\omega) = -90°$$

Step 3

$$\lim_{\omega \to \infty} |G(j\omega)| = 0$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = -270°$$

Step 4 Rationalization

$$G(j\omega) = \frac{K}{j\omega(j\omega+2)(j\omega+10)} \times \frac{(2-j\omega)}{(2-j\omega)} \times \frac{(10-j\omega)}{(10-j\omega)} = \frac{K(20-\omega^2-j12\omega)}{j\omega(4+\omega^2)(100+\omega^2)}$$

$$= -\frac{12K}{(4+\omega^2)(100+\omega^2)} - j\frac{K(20-\omega^2)}{\omega(4+\omega^2)(100+\omega^2)}$$

Step 5

Equate $Re[G(j\omega)] = 0$

$\omega = \pm\infty$ (no distinct values)

Step 6

Equate $Im[G(j\omega)] = 0$

Either, $\omega = \pm\infty$ or, $(20 - \omega^2) = 0$

or, $\omega^2 = 20$

$\therefore$ $\omega = 2\sqrt{5} = 4.47$ rad/sec

This is the frequency at which polar plot intersects the real axis. The intercept is calculated as:

$$G(j\omega) = -\frac{12K}{(\omega^2+4)(\omega^2+100)} - 0$$

Substituting $\omega^2 = 20$; we get,

$$G(j\omega) = -\frac{12K}{(20+4)(20+100)} = -\frac{K}{240} = \left(\frac{K}{240}\right) \angle -180°$$

Hence, the polar plot intersects the negative real axis (*i.e.*, −180°) and the intercept is $\left(\frac{K}{240}\right)$. The again margin is given by p.c.f.

We know that;

P.c.f. $= 2\sqrt{5}$ rad/sec.

Here,

$$|G(S)| = \frac{K}{240}$$

We know that;

$$\text{Gain margin } (K_g) = \frac{1}{|G(S)|} = \frac{1}{\frac{K}{240}} = \frac{240}{K}$$

We have to find out the gain margin in decibel.

so, $(G.M.) = 20\log_{10}(K_g)$

or, $6 = 20\log_{10}\left(\frac{240}{K}\right)$

or, $\frac{6}{20} = \log_{10}\left(\frac{240}{K}\right)$

or, $(10)^{\frac{3}{10}} = \frac{240}{K}$

$\therefore$ $\boxed{K = 120.285}$

Step 7

We have to find out value of K; so that, the system has P.M. = + 45°.

$$\angle G(j\omega) = -90° - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{10}\right)$$

and, $P.M. = 180° + \angle G(j\omega)$

Scanned with CamScanner

or, $45° = 180° - 90° - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{10}\right)$

or, $\tan^{-1}\left(\frac{\omega}{2}\right) + \tan^{-1}\left(\frac{\omega}{10}\right) = 45°$

Taking tan on both sides; we have,

$$\tan\left[\tan^{-1}\left(\frac{\omega}{2}\right) + \tan^{-1}\left(\frac{\omega}{10}\right)\right] = \tan 45°$$

or, $\frac{\frac{\omega}{2} + \frac{\omega}{10}}{1 - \left(\frac{\omega}{2}\right)\left(\frac{\omega}{10}\right)} = 1$

or, $\frac{5\omega + \omega}{10} \times \frac{20}{20 - \omega^2} = 1$

or, $\frac{12\omega}{20 - \omega^2} = 1$

or, $\omega^2 + 12\omega - 20 = 0$

Solving; we get,

$\omega_1 = 1.483$ (valid)

and, $\omega_1 = -13.48$ (negative not possible)

This frequency is the gain crossover frequency because it gives the information about phase margin.

Now, at g.c.f;

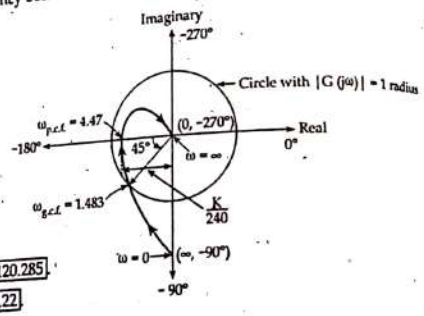
$$|G(j\omega)| = 1$$

or, $\frac{K}{\omega\sqrt{\omega^2 + 4}\sqrt{\omega^2 + 100}} = 1$

Put, $\omega_{g.c.f.} = 1.483$ rad/sec.

or, $\frac{K}{1.483 \times 25.17} = 1$

or, $\boxed{K = 37.22}$



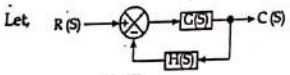
Step 8 Solution

i) Value of K for GM = +6 dB is $\boxed{K = 120.285}$

ii) Value of K for PM = +45° is $\boxed{K = 37.22}$

6.7 NYQUIST PLOT AND STABILITY

Basic concept

Let, 

Let, $G(S) = \frac{N_G(S)}{D_G(S)}$

and, $H(S) = \frac{N_H(S)}{D_H(S)}$

where, N is the numerator and D is the denominator.

We know that; the system is unstable if there is any pole on right half of S-plane.

i) In open loop system;

$$G(S)H(S) = \frac{N_G(S)N_H(S)}{D_G(S)D_H(S)}$$

ii) In characteristics equation

For closed loop system;

$$D(S) = 1 + G(S)H(S) = 1 + \frac{N_G(S)N_H(S)}{D_G(S)D_H(S)} = \frac{D_G(S)D_H(S) + N_G(S)N_H(S)}{D_G(S)D_H(S)}$$

iii) In closed loop transfer function

$$T.F. = \frac{G(S)}{1 + G(S)H(S)} = \frac{\frac{N_G(S)}{D_G(S)}}{1 + \frac{N_G(S)}{D_G(S)} \times \frac{N_H(S)}{D_H(S)}} = \frac{N_G(S)D_H(S)}{D_G(S)D_H(S) + N_G(S)N_H(S)}$$

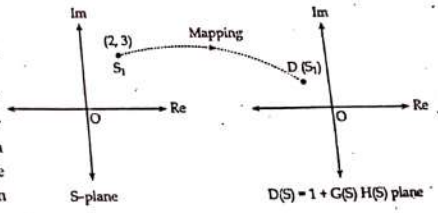
By observing these, we concluded that;

- Poles of open loop system and poles of characteristics equation are same.
- Poles of closed loop and zeros of characteristics equation are same.

6.7.1 Mapping Theory

i) If a contour in S-plane does not contain any poles or zeros, then the contour in D(S) plane will not encircle the origin.

ii) If a contour in S-plane encloses P number of poles and the contour being traced in clockwise direction then the contour in D(S)-plane encircle the origin p-times in counter clockwise direction.



iii) If contour in S-plane encloses Z number of zeros and the contour being traced in clockwise direction, then the contour in D(S)-plane encircle the origin Z-times in clockwise direction.

iv) If S-plane contour encloses P number of poles and Z-number of zeros, then the encirclement of origin in D(S) plane is;

N = (Z − P) times in clockwise direction.

= (P − Z) times in anticlockwise direction.

6.7.2 Nyquist Path, Nyquist Contour and Nyquist Plot

For stability analysis, consider a closed loop contour in S-plane, which enclose the entire right half of s-plane. That close contour consists of entire imaginary axis from $j\omega = -\infty$ to $j\omega = +\infty$, and a semicircle of infinite radius in right half of S-plane. Such a closed contour is known as Nyquist path.

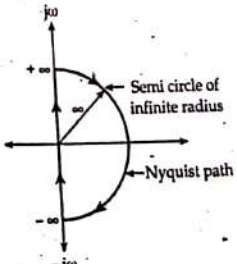


Figure: (i) Nyquist path in S-plane

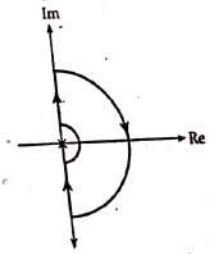


Figure: (ii) Nyquist contour for G (S) = $\frac{1}{S}$

Scanned with CamScanner

Nyquist path should not pass through any poles and zeros of characteristics equation. If Nyquist path passes through the location of poles and zeros, mapping of that point will become indeterminate. So, such points are avoided by taking detour around it.

Hence, if contour in S-plane encircles all the region on right half s-plane of imaginary axis excluding any poles and zeros of characteristics equation if present in the imaginary axis, then it is called **Nyquist contour.**

For example: Nyquist contour for $G(S) = \frac{1}{S}$

The mapping of Nyquist contour in to G(S)H(S)-plane is called **Nyquist plot.**

6.7.3 Stability Criteria

The number of right half plane zeros of characteristics equation [D(S) = 1 + G(S)H(S)] is equal to the number of poles of D(S) in the right half of S-plane plus the number of clockwise encirclements of the origin of [D(S) = 1 + G(S)H(S)]; [OR (−1, 0) of G(S)H(S)] by the corresponding closed curve.

i.e., $Z = N + P$

where, P = Open loop poles in right half.
= Poles of characteristics equation having + ve real part.
Z = Zeros of characteristic equation having + ve real part.

i) The feedback system is stable if and only if Z = 0.
i.e., no zeros of characteristics equation = Number of poles of feedback system (closed loop)
ii) If Z ≠ 0; the system is unstable.
iii) If Nyquist plot in G(S)H(S) plane encloses (−1, 0) it becomes unstable, if it doesn't enclose (−1, 0); then, check
Z = N + P = 0; for stability

6.7.4 Steps to Sketch Nyquist Plot

i) Sketch polar plot.
ii) Draw mirror image of plot symmetrical to real axis.
iii) Join tow ends (if any) of plot by hypothetical curve of radius R = ∞
iv) Sketch Nyquist contour
v) Check stability criteria

EXAMPLE 6.9

Determine the close loop stability of control system whose open loop transfer function is $G(S)H(S) = \frac{K}{S(1+ST)}$.

Solution:

Step 1

Put, S = jω; then,

$$G(j\omega) = \frac{K}{j\omega(1+j\omega T)} = \frac{K}{\omega\sqrt{1+\omega^2T^2}} \angle -90° - \tan^{-1}(\omega T)$$

Step 2

$$\lim_{\omega \to 0} |G(j\omega)| = \infty$$

$$\lim_{\omega \to 0} \angle G(j\omega) = -90°$$

Step 3

$$\lim_{\omega \to \infty} |G(j\omega)| = 0$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = -180°$$

Step 4 Rationalization

$$G(j\omega) = \frac{K}{j\omega(1+j\omega T)} \times \frac{(1-j\omega T)}{(1-j\omega T)} = \frac{K(1-j\omega T)}{j\omega\left(1+\omega^2T^2\right)} = \frac{K}{j\omega\left(1+\omega^2T^2\right)} - \frac{Kj\omega T}{j\omega\left(1+\omega^2T^2\right)}$$

$$= -\frac{KT}{\left(1+\omega^2T^2\right)} - j\frac{K}{\left(1+\omega^2T^2\right)}$$

Step 5

Equating $Re\{G(j\omega)\} = 0$

$\omega = \pm\infty$ (no distinct value)

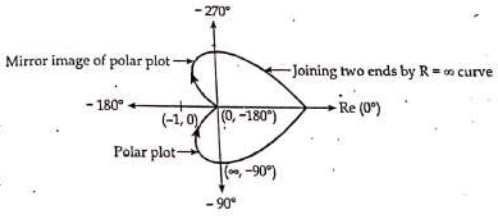
Step 6

Equating $Im\{G(j\omega)\} = 0$

$\omega = \pm\infty$ (no distinct value)

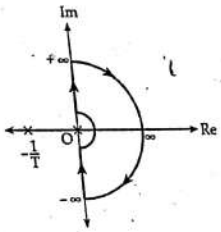
The polar plot doesn't cross any axis.

Step 7 Sketch polar plot and Nyquist



Step 8

Nyquist contour for $G(S) = \frac{1}{S(1+ST)}$



Here,

P = Open loop poles inside Nyquist contour = 0

N = Number of encirclement that encloses (−1, 0) = 0

Check for stability

$Z = N + P$

or, $Z = 0 + 0$

∴ Z = 0; so, the system is stable.

EXAMPLE 6.10

Use Nyquist criteria to determine stability of the system. $G(S) = \frac{K(1-S)}{(1+S)}$ and H(S) = 1

Solution:

Step 1

Put, S = jω; then,

$$G(j\omega) = \frac{K(1-j\omega)}{(1+j\omega)} = \frac{K\sqrt{1+\omega^2}}{\sqrt{1+\omega^2}} \angle -\tan^{-1}(\omega) + \tan^{-1}(-\omega) = K\angle -2\tan^{-1}(\omega)$$

Scanned with CamScanner

Step 2

$$\lim_{\omega \to 0} |G(j\omega)| = K$$

$$\lim_{\omega \to 0} \angle G(j\omega) = 0$$

Step 3

$$\lim_{\omega \to \infty} |G(j\omega)| = K$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = -180°$$

Step 4 Rationalization

$$G(j\omega) = \frac{K(1-j\omega)}{(1+j\omega)} \times \frac{1-j\omega}{1-j\omega} = \frac{K(1-j\omega)^2}{(1+\omega^2)} = \frac{K(1-\omega^2)}{(1+\omega^2)} - j\frac{2K\omega}{(1+\omega^2)}$$

Step 5

Equating $Re\{G(j\omega)\} = 0$

$$\frac{K(1-\omega^2)}{(1+\omega^2)} = 0$$

Either, $\omega = \pm\infty$ or, $(1-\omega^2) = 0$

or, $\omega^2 = 1$

$\therefore \quad \omega = 1$ rad/sec.

This is the frequency at which polar plot intersects imaginary axis. The intercept can be calculated by;

$$G(j\omega) = 0 - j\frac{2K\omega}{(1+\omega^2)}$$

Put, $\omega = 1$; then,

$$G(j\omega) = -j\left(\frac{2K}{2}\right) = -jK = K\angle -90°$$

Step 6

Equating $Im\{G(j\omega)\} = 0$

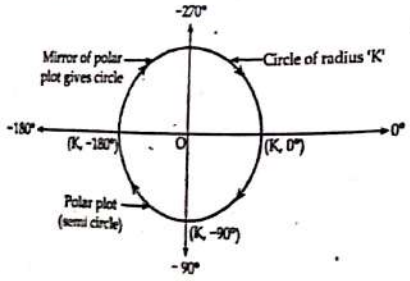
$$-j\frac{2K\omega}{(1+\omega^2)} = 0$$

Either, $\omega = 0$ or, $\omega = \pm\infty$ (no distinct values)

Hence, polar plot do not intersects on real axis.

Step 7

The polar plot modifies into Nyquist plot as;



Step 8

Nyquist contour for $G(S) = \frac{K(1-S)}{(1+S)}$

Open loop poles = Number of poles of characteristics equation

= P

= Number of poles inside Nyquist contour

Now,

N = Number of encirclement that encloses (−1, 0)

Case I

When $K < 1$

P = 0 (Number of poles inside contour); and Nyquist plot does not enclose (−1, 0)

so, $N = 0$

and, $Z = N + P$

or, $Z = 0 + 0$

or, $Z = 0$; hence, stable

Case II

When $K > 1$

It encloses (−1, 0); so,

$N = 1$

P = Number of poles inside contour = 0

Now,

$Z = N + P = 1 \neq 0$; hence, unstable

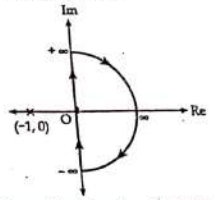


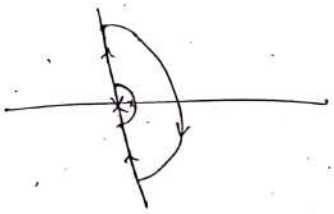
Figure: Nyquist contour of radius R = ∞

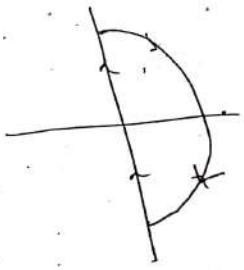
Case III

When $K = 1$

The system is marginally stable.

Nyquist contour.





If Origin is a pole and real value are poles.

If origin and +ve real value are not poles

Scanned with CamScanner

6.8. SOLVED PROBLEMS

1. **Using Nyquist criteria, determine the stability of the feedback system whose OLTF is given by $G(S)H(S) = \frac{1}{S(1+2S)(1+S)}$.** [2073 Chaitra]

Solution:

Step 1 Draw polar plot

Put, $S = j\omega$; then,

$$G(j\omega)H(j\omega) = \frac{1}{j\omega(1+2j\omega)(1+j\omega)} = \frac{1}{\omega\sqrt{1+4\omega^2}\sqrt{1+\omega^2}} \angle -90° - \tan^{-1}(\omega) - \tan^{-1}(2\omega)$$

Step 2

$$\lim_{\omega \to 0} |G(j\omega)H(j\omega)| = \infty$$

$$\lim_{\omega \to 0} \angle[G(j\omega)H(j\omega)] = -90°$$

Step 3

$$\lim_{\omega \to \infty} |G(j\omega)H(j\omega)| = 0$$

$$\lim_{\omega \to \infty} \angle[G(j\omega)H(j\omega)] = -270°$$

Step 4 Rationalization

$$G(j\omega)H(j\omega) = \frac{1}{j\omega(1+2j\omega)(1+j\omega)} \times \frac{1-j\omega}{1-j\omega} \times \frac{1-j\cdot 2\omega}{1-j\cdot 2\omega}$$

$$= \frac{1-2\omega^2-j\cdot 3\omega}{j\omega(1+4\omega^2)(1+\omega^2)} = -\frac{j(1-2\omega^2)}{\omega(1+\omega^2)(1+4\omega^2)} - \frac{3}{(1+\omega^2)(1+4\omega^2)}$$

$$= j\frac{2\omega^2-1}{\omega(1+\omega^2)(1+4\omega^2)} - \frac{3}{(1+\omega^2)(1+4\omega^2)}$$

Step 5

Equating $\text{Im}\{G(j\omega)\} = 0$

Either, $\omega = \pm\infty$ or, $2\omega^2 - 1 = 0$

or, $\omega^2 = \frac{1}{2}$

$\therefore \omega = \frac{1}{\sqrt{2}}$

This is the frequency at which polar plot intersects real axis. The intercept is calculated by;

$$G(j\omega)H(j\omega) = 0 - \frac{3}{(1+\omega^2)(1+4\omega^2)}$$

Substituting $\omega^2 = \frac{1}{2}$; we have,

$$G(j\omega)H(j\omega) = -\frac{3}{\left(1+\frac{1}{2}\right)\left(1+4\times\frac{1}{2}\right)} = -\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)\angle -180° = 0.66\angle -180°$$

Step 6

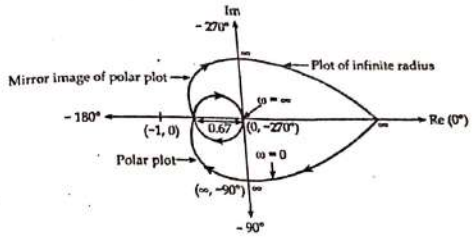
Equating $\text{Re}\{G(j\omega)\} = 0$; then,

$\omega = \pm\infty$

No distinct values so polar plot do not intersect in imaginary axis.

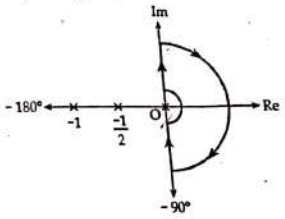
Step 7

The polar plot modifies into Nyquist plot as;



Step 8

Nyquist contour for $G(j\omega)H(j\omega) = \frac{1}{S(1+2S)(1+S)}$ is;



P = Number of poles inside Nyquist contour = 0

N = Number of encirclement that encloses (−1, 0) = 0

Now,

$Z = N + P = 0 + 0 = 0$

$\therefore Z = 0$

Hence, the system is stable.

2. **Construct the polar plot of unity feedback system with $G(S) = \frac{K}{S(S+1)(0.1S+1)}$ then, upgrade the plot to make it Nyquist plot. Hence find range of K for stable operation.** [2071 Shrawan]

Solution:

Step 1

Put, $S = j\omega$; then,

$$G(j\omega) = \frac{K}{j\omega(j\omega+1)(0.1j\omega+1)} = \frac{K}{\omega\sqrt{1+\omega^2}\sqrt{1+0.01\omega^2}} \angle -90° - \tan^{-1}(\omega) - \tan^{-1}(0.1\omega)$$

Step 2

$$\lim_{\omega \to 0} |G(j\omega)| = \infty$$

$$\lim_{\omega \to 0} \angle G(j\omega) = -90°$$

Scanned with CamScanner

Step 3

$$\lim_{\omega \to \infty} |G(j\omega)| = 0$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = -270°$$

Step 4 Rationalization

$$G(j\omega) = \frac{K}{j\omega(1+j\omega)(1+j0.1\omega)} \times \frac{1-j0.1\omega}{1-j0.1\omega} \times \frac{1-j\omega}{1-j\omega}$$

$$= \frac{K(1-0.1\omega^2-j1.1\omega)}{j\omega(1+\omega^2)(1+0.01\omega^2)} = -\frac{jK(1-0.1\omega^2)}{\omega(1+\omega^2)(1+0.01\omega^2)} - \frac{1.1K}{(1+\omega^2)(1+0.01\omega^2)}$$

Step 5

Equating $Im\{G(j\omega)\} = 0$

$\omega = \pm\infty$

and, $1-0.1\omega^2 = 0$

or, $\omega^2 = \frac{1}{0.1} = 10$

or, $\omega = \sqrt{10}$

$\therefore \omega = 3.162$ rad/sec.

This is the frequency where polar plot intersects real axis, and the intercept is calculated by;

$$G(j\omega) = 0 - \frac{1.1K}{(1+\omega^2)(1+0.01\omega^2)}$$

Put, $\omega^2 = 10$, we get

$$G(j\omega) = -\frac{1.1K}{(1+10)(1+0.01\times 10)} = -\frac{1.1K}{11\times 1.1} = -\frac{K}{11} = \left(\frac{K}{11}\right) \angle -180°$$

Step 6

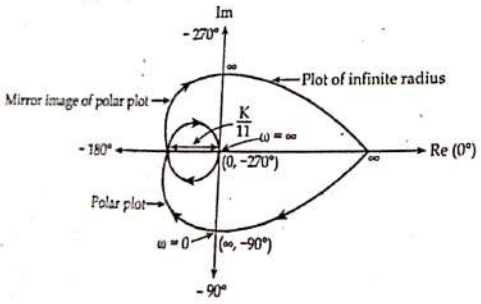
Equating $Re\{G(j\omega)\} = 0$;

$\omega = \pm\infty$

No distinct values; so, plot doesn't intersect on imaginary axis.

Step 7

The polar plot modifies into Nyquist plot as;

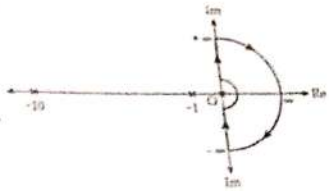


Step 8

The Nyquist contour for $G(S) = \frac{K}{S(S+1)(0.1S+1)}$ is;

P = Number of poles inside Nyquist contour = 0

N = Number of encirclement that encloses (−1, 0)



Case I

When $\left(\frac{K}{11}\right) > 1$, then,

N = Number of entirclement that encloses (−1, 0) = 0

$Z = N + P = 2 + 0 = 2 \neq 0$

The system is unstable.

Case II

When $\left(\frac{K}{11}\right) < 1$, then,

N = Number of circlement that encloses (−1, 0) = 0

$Z = N + P = 0$,

The system is stable.

Case III

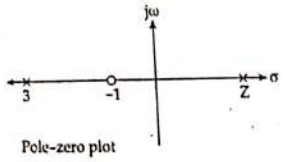
When $\left(\frac{K}{11}\right) = 1$; then,

The system is marginally stable.

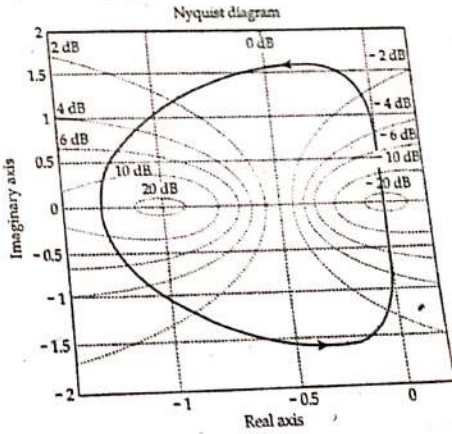
Range of stability is $\left[-\infty < \left(\frac{K}{11}\right) < 1\right]$.

i.e., range of K is $[-\infty < K < 11]$.

3. State the Nyquist stability criteria for negative feedback control system. Using this concept determine whether the following system represented by figure below is stable. [2071 Chaitra]



Pole-zero plot



Solution:

For the first part

See the definition part 6.7.3

Scanned with CamScanner

For the second part

Step 1

From question above; we have,

Pole at $S_1 = -3$

Pole at $S_2 = 2$

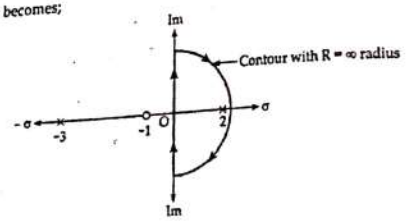
Zero at $S_3 = -1$

Therefore, transfer function is given by;

$$T.F. = \frac{(S+1)}{(S-2)(S+3)}$$

Step 2

The Nyquist contour becomes;



Here,

P = Number of poles inside Nyquist contour = 1

Nyquist contour should not contain any poles and zeros in the right half of S-plane. If Nyquist contour encloses it, mapping of that point will become indeterminate;

and, N = Number of encirclement that encloses (−1, 0)

= which is always + ve for clockwise direction

$\therefore$ Z = N + P = 1 + (+ve value) = Greater than one $\neq 0$

Therefore, the system is **unstable**.

4. Sketch Nyquist plot for $G(S) = \frac{(S+2)}{(S+1)(S-1)}$ and comment on stability. [2070 Chaitra]

Solution:

Step 1 Draw polar plot

Put, S = jω; then,

$$G(j\omega) = \frac{(j\omega+2)}{(j\omega+1)(j\omega-1)} = \frac{(j\omega+2)}{-\omega^2-1} = -\frac{(j\omega+2)}{\omega^2+1} \angle \tan^{-1}\left(\frac{\omega}{2}\right) = \frac{\sqrt{4+\omega^2}}{1+\omega^2} \angle \tan^{-1}\left(\frac{\omega}{2}\right) - 180°$$

Step 2

$$\lim_{\omega \to 0} |G(j\omega)| = 2$$

$$\lim_{\omega \to 0} \angle G(j\omega) = -180°$$

Step 3

$$\lim_{\omega \to \infty} |G(j\omega)| = 0$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = 90° - 180° = -90°$$

Step 4 Separating real and imaginary part

$$G(j\omega) = -\frac{(j\omega+2)}{\omega^2+1} = -\frac{2}{\omega^2+1} - j\frac{\omega}{\omega^2+1}$$

Step 5

Equating $Re\{G(j\omega)\} = 0$;

$$-\frac{2}{\omega^2+1} = 0$$

i.e., $\omega = \pm\infty$

No distinct value so, polar plot do not intersects imaginary axis.

Step 6

Equating $Im\{G(j\omega)\} = 0$;

$$-j\frac{\omega}{\omega^2+1} = 0$$

Either, $\omega = \pm\infty$ or, $\omega = 0$

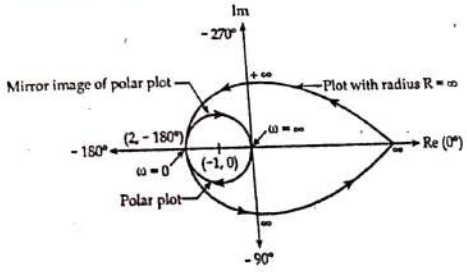
Now, substituting $\omega = 0$ on real axis; we get,

$$G(j\omega) = -\frac{2}{\omega^2+1} - 0 = -\frac{2}{0+1} = -2 = 2\angle -180°$$

i.e., polar plot intersects negative real axis.

Step 7

The polar plot modifies into Nyquist plot as below:



Step 8

The Nyquist contour of $G(S) = \frac{(S+2)}{(S+1)(S-1)}$

Here,

P = Number of poles in Nyquist contour = 1

N = Number of encirclement that encloses (−1, 0) is, = 2

The encirclement here is in anticlockwise direction.

so, N = (P − Z)

or, Z = P − N

or, Z = 1 − 2

or, Z = −1 ≠ 0

Hence, the system is **unstable.**

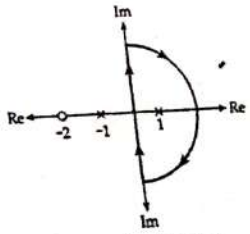


Figure: Nyquist contour

Scanned with CamScanner

5. Examine the closed loop stability of a system applying Nyquist criterion whose open loop transfer function is given by $G(S)H(S) = \frac{50}{(S+1)(S+2)}$. [2067 Ashadh]

Solution:

Step 1 Sketch polar plot

Put, $S = j\omega$; then,

$$G(j\omega)H(j\omega) = \frac{50}{(j\omega+1)(j\omega+2)} = \frac{50}{\sqrt{\omega^2+1}\sqrt{\omega^2+4}} \angle -\tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{2}\right)$$

Step 2

$$\lim_{\omega \to 0} |G(j\omega)| = \frac{50}{2} = 25$$

$$\lim_{\omega \to 0} \angle G(j\omega) = 0°$$

Step 3

$$\lim_{\omega \to \infty} |G(j\omega)| = 0$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = -180°$$

Step 4 Rationalization

$$G(j\omega)H(j\omega) = \frac{50}{(2+j\omega)(1+j\omega)} \times \frac{(1-j\omega)}{(1-j\omega)} \times \frac{(2-j\omega)}{(2-j\omega)} = \frac{50(2-\omega^2-j3\omega)}{(4+\omega^2)(1+\omega^2)}$$

$$= \frac{50(2-\omega^2)}{(4+\omega^2)(1+\omega^2)} - j\frac{150\omega}{(4+\omega^2)(1+\omega^2)}$$

Step 5

Equating $Re\{G(j\omega)H(j\omega)\} = 0$;

$$\frac{50(2-\omega^2)}{(4+\omega^2)(1+\omega^2)} = 0$$

Either, $\omega = \pm\infty$ or, $2-\omega^2 = 0$

or, $\omega^2 = 2$

$\therefore \omega = \sqrt{2}$

This is distinct value; hence, plot intersects imaginary axis. The intercept can be calculated as;

$$G(j\omega)H(j\omega) = 0 + j\frac{150 \times \sqrt{2}}{(4+2)(1+2)}$$

$$= -j\left(\frac{25\sqrt{2}}{3}\right) = \left(\frac{25\sqrt{2}}{3}\right)\angle(-j) = \left(\frac{25\sqrt{2}}{3}\right)\angle -90°$$

$$= 11.78\angle -90°$$

Step 6

Equating $Im\{G(j\omega)H(j\omega)\} = 0$;

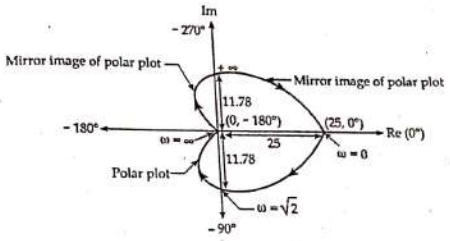
$$-j\frac{150\omega}{(4+\omega^2)(1+\omega^2)} = 0$$

Either, $\omega = \pm\infty$ or, $\omega = 0$

which is not a distinct value; hence, polar plot doesn't intersects on real axis.

Step 7

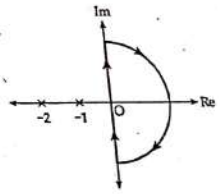
The polar plot modifies in to Nyquist plot as;



Step 8

Here, the Nyquist contour;

$$G(S)H(S) = \frac{50}{(S+1)(S+2)}$$



Let, N = Number of encirclement that encloses (-1, 0) = 0

P = Number of poles on Nyquist contour = 0

Now, $Z = N + P = 0 + 0 = 0$

$\therefore$ $Z = 0$; so, the system is stable.

6. For a unity feedback system with $G(S) = \frac{250K}{S(S+50)(S+5)}$, select the value of K so that the steady state error for a ramp input is 10%. At this value of K, construct the Bode plot. Hence, determine appropriate G.M., P.M., P.C.F. and G.C.F. [2072 Kartik]

Solution:

Since, the input is ramp,

so, $K_V = \lim_{S \to 0} S \times G(S)H(S) = \lim_{S \to 0} S \times \frac{250K}{S(S+50)(S+5)} = \frac{250K}{5 \times 50} = K$

Now,

Steady state error $(e_{ss}) = \frac{1}{K_V}$

or, $10\% = \frac{1}{K_V}$

or, $K_V = \frac{1}{0.1}$

$\therefore$ $K = K_V = 10$ sec.$^{-1}$

Now, drawing Bode plot as shown below.

Step 1

$$T.F. = G(S) = \frac{250K}{S(S+50)(S+5)} = \frac{250 \times 10}{5 \times 50 \times S\left(1+\frac{S}{5}\right)\left(1+\frac{S}{50}\right)} = \frac{10}{S(1+0.2S)(1+0.02S)}$$

Put, $S = j\omega$;

$$G(j\omega) = \frac{10}{j\omega(1+0.2j\omega)(1+0.02j\omega)}$$

Step 2 Determining corner frequencies

For poles,

Scanned with CamScanner

$\omega_1 = \frac{1}{0.2} = 5$ rad/sec. (Smallest)

and, $\omega_2 = \frac{1}{0.02} = 50$ rad/sec.

Step 3

Starting frequency $(\omega) = \frac{1}{10} \times 5 = 0.5 \approx 0.1$ (approximate to nearest decade)

Step 4

The transfer function is type-one system.

so, $N = 1$

$\omega = 0.1$ rad/sec.

and, $K = 10$

Staring magnitude $= 20\log_{10} K - 20N\log_{10}\omega = 20\log_{10} 10 - 20 \times 1\log_{10}(0.1)$

$= 20 - 20 \times (-1) = 40$ dB

Step 5

From $\omega = 0.1$ to $\omega = 5$ rad/sec.;

Slope $= -20$ dB/dec.

From $\omega = 5$ to $\omega = 50$ rad/sec.;

Slope $= -20 - 20 = -40$ dB/dec.

From $\omega = 50$ to $\omega =$ higher;

Slope $= -40 + (-20) = -60$ dB/dec.

Step 6 Phase plot

$\phi_{Total} = \angle G(j\omega) = -90° - \tan^{-1}(0.2\omega) - \tan^{-1}(0.02\omega)$

ω	ϕ_{Total}	ω	ϕ_{Total}
0.1	− 91.26°	20	− 187.8°
0.2	− 92.52°	50	− 219.29°
0.5	− 96.28°	100	− 240.6°
1	− 102.45°	500	− 263.71°
10	− 164.74°	1000	− 266.85°

Step 7 Solutions from graph

Plotting magnitude and phase plot

p.c.f = Frequency at which phase plot intersects − 180° line

= 16 rad/sec.

g.c.f = Frequency at which magnitude plot intersects 0 dB line

= 6.8 rad/sec.

P.M. = Margin in phase given by g.c.f.

$= -150° - (-180°)$

$= 180° - 150°$

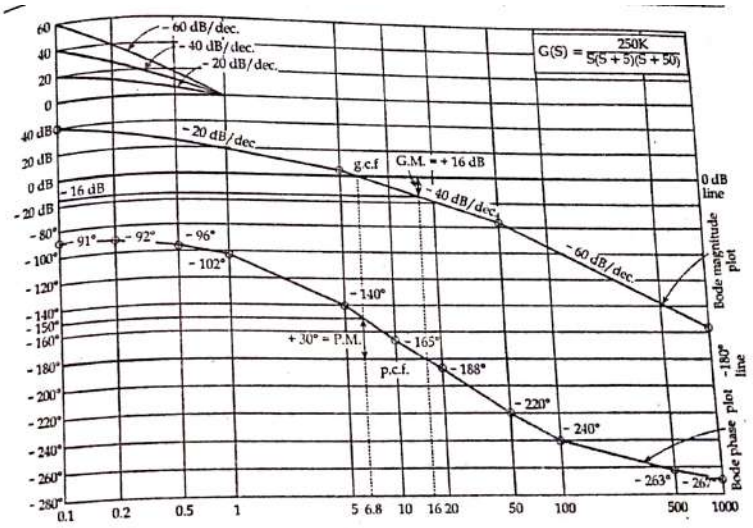
$= +30°$

G.M. = Margin in gain given by p.c.f

$= 0 - (-16)$

$= 16$ dB

Since, both gain and phase margin are positive, the system is **stable.**



7. The open loop transfer function of a control system is $G(S)H(S) = \frac{(4S+1)}{S^2(S+1)(2S+1)}$. Using Nyquist criteria, determine the open loop and closed loop stability of this system. **[2074 Ashwin]**

Solution:

Step 1 Draw polar plot

Put $S = j\omega$;

$$G(j\omega)H(j\omega) = \frac{(1 + j\cdot 4\omega)}{-\omega^2(1 + j\omega)(1 + j\cdot 2\omega)} = \frac{\sqrt{1 + 16\omega^2}}{\omega^2\sqrt{1 + \omega^2}\sqrt{1 + 4\omega^2}} \angle -180° - \tan^{-1}\omega - \tan^{-1}2\omega + \tan^{-1}4\omega$$

Step 2

$$\lim_{\omega \to 0} |G(j\omega)H(j\omega)| = \infty$$

$$\lim_{\omega \to 0} \angle G(j\omega)H(j\omega) = -180°$$

Step 3

$$\lim_{\omega \to \infty} |G(j\omega)H(j\omega)| = 0$$

$$\lim_{\omega \to \infty} \angle G(j\omega)H(j\omega) = -180° - 90° - 90° + 90° = -270°$$

Step 4 Rationalization

$$G(j\omega)H(j\omega) = \frac{(1 + j\cdot 4\omega)}{-\omega^2(1 + j\omega)(1 + j\cdot 2\omega)} = -\frac{(1 + j\cdot 4\omega)}{\omega^2(1 - 2\omega^2 + j\cdot 3\omega)} \times \frac{(1 - 2\omega^2 - j\cdot 3\omega)}{(1 - 2\omega^2 - j\cdot 3\omega)}$$

$$= -\frac{(1 + j\cdot 4\omega)(1 - 2\omega^2 - j\cdot 3\omega)}{\omega^2\left[(1 - 2\omega^2)^2 - j^2(3\omega)^2\right]} = -\frac{\left[1 + 10\omega^2 + j\omega(1 - 8\omega^2)\right]}{\omega^2(1 + 5\omega^2 + 4\omega^4)}$$

Scanned with CamScanner

$$= -\frac{1+10\omega^2}{\omega^2(1+5\omega^2+4\omega^4)} - j\left[\frac{1-8\omega^2}{\omega(1+5\omega^2+4\omega^4)}\right]$$

Step 5

Equate $Re\{G(j\omega)H(j\omega)\} = 0$

Either, $\omega = \pm\infty$

or, $1 + 10\omega^2 = 0$

or, $10\omega^2 = -1$

or, $10\omega^2 = j^2$

$\therefore \quad \omega = \pm\frac{j}{\sqrt{10}} = \pm 0.3162j$

The value of ω here comes is in imaginary form and not a distinct value. So, we can conclude that the polar plot does not intersect imaginary axis.

Step 6

Equate $Im\{G(j\omega)H(j\omega)\} = 0$

Either, $\omega = \pm\infty$

or, $1 - 8\omega^2 = 0$

or, $8\omega^2 = 1$

or, $\omega^2 = \frac{1}{8}$

$\therefore \quad \omega = \pm\frac{1}{2\sqrt{2}} = 0.3535$ rad/sec.

This frequency is in distinct form and hence we can say that polar plot intersects the real axis.

The intercept is calculated by;

$$G(j\omega)H(j\omega) = -\frac{1+10\omega^2}{\omega^2(1+5\omega^2+4\omega^4)} - 0$$

Substituting $\omega^2 = \frac{1}{8}$; we get,

$$G(j\omega)H(j\omega) = -\frac{1+\frac{10}{8}}{\frac{1}{8}\left(1+5\times\frac{1}{8}+4\times\frac{1}{64}\right)} = -\frac{32}{3} = -10.667\angle -180°$$

Step 7

The polar plot now modifies into Nyquist plot as;

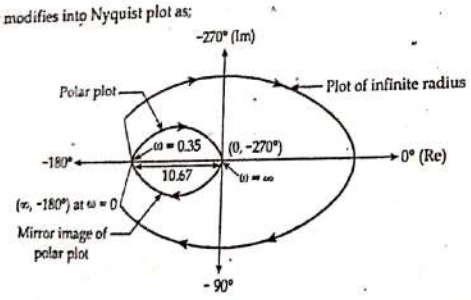


Figure: Nyquist plot

Step 8

Nyquist contour for $G(j\omega)H(j\omega) = \frac{(4S+1)}{S^2(S+1)(2S+1)}$ is;

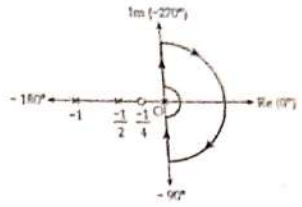
where, P is the number of poles inside Nyquist contour = 0

N is the number of encirclement that encloses $(-1, 0j) = 2$

Now,

$$Z = N + P = 2 + 0 = 2 \neq 0$$

Hence, two roots of the characteristics equation lies in the right half of S-plane since $Z \neq 0$, so, the system is unstable.



8. Draw Bode plot for the system with transfer function $G(S) = \frac{20S+200}{(S^2+4S+25)(S^2+40S)}$.

Determine gain margin, phase margin and comment on stability of the system according to you plot. [2074 Ashwin]

Solution:

Here,

$$G(S) = \frac{200\left(1+\frac{20S}{200}\right)}{(S\times 25\times 40)\left(1+\frac{4}{25}S+\frac{S^2}{25}\right)\left(1+\frac{S}{40}\right)}$$

$$= \frac{200(1+0.1S)}{1000S(1+0.025S)\left[1+\left(\frac{4}{25}\right)S+\left(\frac{S}{5}\right)^2\right]}$$

$$= \frac{0.2(1+0.1S)}{S(1+0.025S)\left[1+\left(\frac{4}{25}\right)S+\left(\frac{S}{5}\right)^2\right]}$$

For the further steps

Proceed as example 6.5

Scanned with CamScanner

Chapter 7

Performance Specifications and Compensation Design

7.1. Time domain specification 204
7.1.1 Rise time, peak time, delay time, settling time and maximum overshoot 204
7.1.2 Static error coefficient 205
7.1.2.1 Steady state error (e_{ss}) 205
7.1.2.2 Static error coefficient 207
7.1.3 Steady state error for different types of system 231
7.2 Frequency domain specification 231
7.2.1 Gain margin and phase margin 231
7.3. Compensation and it's type 231
7.3.1 Compensation 232
7.3.2 Types of compensation 232
7.4 Design of compensation using Bode's plot 232
7.4.1 Phase lead compensation 241
7.4.2 Phase lag compensation (Pi controller) 244
7.5. Design of compensation using root locus 246
7.5.1 Design step for lead compensator 252
7.5.2 Design step for lag compensator 256
7.6 PID controllers 258
7.6.1 Controllers (Compensator) 258
7.6.1.1 Proportional controller (P – controller) 259
7.6.1.2 Derivative controller (D–controller) 260
7.6.1.3 Integral controller (I–controller) 260
7.6.1.4 PD controller (Lead compensator) 261
7.6.1.5 PI–controller (Lag compensator) 262
7.6.1.6 PID controller

7.1. TIME DOMAIN SPECIFICATION

Already discussed in definition part 3.4

7.1.1 Rise Time, Peak time, Delay Time, Settling Time and Maximum Overshoot

Already discussed in definition part 3.4.4

Scanned with CamScanner

i) Delay time (t_d)

ii) Rise time $(t_r) = \dfrac{\pi - \tan^{-1}\left(\dfrac{\sqrt{1-\zeta^2}}{\zeta}\right)}{\omega_n\sqrt{1-\zeta^2}}$

iii) Peak time $(t_p) = \dfrac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$

iv) Maximum overshoot $(M_p) = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)} \times 100\%$

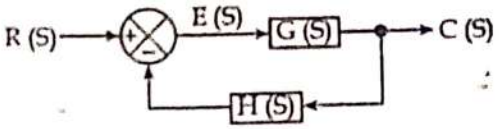
v) Settling time $(t_s) = \dfrac{4}{\zeta\omega_n}$

7.1.2 Static Error Coefficient

7.1.2.1 Steady State Error (e_{ss})

The steady state error is the difference between the input and output of the system during steady state. For accuracy the steady state error should be minimum.

Consider a closed loop system;



$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)}$$

From the figure; we have,

$$C(S) = E(S)G(S)$$

where, E(S) is the error signal.

so, $\dfrac{E(S)G(S)}{R(S)} = \dfrac{G(S)}{1+G(S)H(S)}$

or, $\dfrac{E(S)}{R(S)} = \dfrac{1}{1+G(S)H(S)}$

or, $E(S) = \dfrac{R(S)}{1+G(S)H(S)}$

$\therefore$ $\boxed{E(S) = \dfrac{R(S)}{1+G(S)H(S)}}$; for -ve feedback system

and, $\boxed{E(S) = \dfrac{R(S)}{1-G(S)H(S)}}$; for +ve feedback system

Steady state error can be found by applying final value theorem.

$$e_{ss} = \lim_{t \longrightarrow \infty} e(t) = \lim_{S \longrightarrow 0} SE(S)$$

$\therefore$ $\boxed{e_{ss} = \lim_{S \longrightarrow 0} S\left[\dfrac{R(S)}{1+G(S)H(S)}\right]}$

Hence, steady state error depends upon input and open loop transfer function when H(S) = 1.

7.1.2.2 Static Error Coefficient

i) **Static position error constant (or coefficient) K_p**

We know that;

$$e_{ss} = \lim_{S \longrightarrow 0} \frac{SR(S)}{1+G(S)H(S)}$$

Scanned with CamScanner

For unit step input,

$$R(S) = \frac{1}{S}$$

$$\therefore \quad e_{ss} = \lim_{S \to 0} S \times \frac{1}{S} \times \frac{1}{1 + G(S)H(S)}$$

$$\text{or,} \quad e_{ss} = \lim_{S \to 0} \frac{1}{1 + G(S)H(S)}$$

$$\text{or,} \quad e_{ss} = \frac{1}{1 + \lim_{S \to 0} G(S)H(S)}$$

$$\therefore \quad \boxed{e_{ss} = \frac{1}{1 + K_p}}$$

where, $\boxed{K_p = \lim_{S \to 0} G(S)H(S)}$

II) Static Velocity error constant (or coefficient) K_v

We know that;

$$e_{ss} = \lim_{S \to 0} \frac{SR(S)}{1 + G(S)H(S)}$$

For ramp input;

$$R(S) = \frac{1}{S^2}$$

$$\text{or,} \quad e_{ss} = \lim_{S \to 0} S \times \frac{1}{S^2} \times \frac{1}{1 + G(S)H(S)}$$

$$\text{or,} \quad e_{ss} = \frac{1}{\lim_{S \to 0} SG(S)H(S)}$$

$$\therefore \quad \boxed{e_{ss} = \frac{1}{K_v}}$$

where, $\boxed{K_v = \lim_{S \to 0} SG(S)H(S)}$

III) Static acceleration error constant (or coefficient) K_a

We know that;

$$e_{ss} = \lim_{S \to 0} \frac{SR(S)}{1 + G(S)H(S)}$$

For parabolic input; we have,

$$R(S) = \frac{1}{S^3}$$

$$\text{or,} \quad e_{ss} = \lim_{S \to 0} S \times \frac{1}{S^3} \times \frac{1}{1 + G(S)H(S)} = \lim_{S \to 0} \frac{1}{S^2[1 + G(S)H(S)]} = \lim_{S \to 0} \frac{1}{S^2 + S^2G(S)H(S)}$$

$$= \frac{1}{\lim_{S \to 0} S^2G(S)H(S)}$$

$$\therefore \quad \boxed{e_{ss} = \frac{1}{K_a}}$$

where, $\boxed{K_a = \lim_{S \to 0} S^2G(S)H(S)}$

7.1.3 Steady State Error for Different Types of System

I) For type–zero system

Let, $G(S)H(S) = \dfrac{K(1 + ST_a)(1 + ST_b)}{(1 + ST_1)(1 + ST_2)}$

$$K_p = \lim_{S \to 0} G(S)H(S) = K$$

$$\therefore \quad e_{ss} = \frac{1}{1 + K_p} = \frac{1}{1 + K}$$

$$K_v = \lim_{S \to 0} SG(S)H(S) = 0$$

$$\therefore \quad e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty$$

$$K_a = \lim_{S \to 0} S^2G(S)H(S) = 0$$

$$\therefore \quad e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

II) For type–I system

Let, $G(S)H(S) = \dfrac{K(1 + ST_a)(1 + ST_b)}{S(1 + ST_1)(1 + ST_2)}$

Now,

$$K_p = \lim_{S \to 0} G(S)H(S) = \frac{1}{0} = \infty$$

$$\therefore \quad e_{ss} = \frac{1}{1 + K_p} = 0$$

$$K_v = \lim_{S \to 0} SG(S)H(S) = K$$

$$\therefore \quad e_{ss} = \frac{1}{K_v} = \frac{1}{K}$$

and, $K_a = \lim_{S \to 0} S^2G(S)H(S) = 0$

$$\therefore \quad e_{ss} = \frac{1}{K_a} = \infty$$

III) For type–II system

Let, $G(S)H(S) = \dfrac{K(1 + ST_a)(1 + ST_b)}{S^2(1 + ST_1)(1 + ST_2)}$

Now,

$$K_p = \lim_{S \to 0} G(S)H(S) = \infty$$

$$\therefore \quad e_{ss} = \frac{1}{1 + K_p} = \frac{1}{\infty} = 0$$

$$K_v = \lim_{S \to 0} SG(S)H(S) = \infty$$

$$\therefore \quad e_{ss} = \frac{1}{K_v} = \frac{1}{\infty} = 0$$

and, $K_a = \lim_{S \to 0} S^2G(S)H(S) = K$

$$\therefore \quad e_{ss} = \frac{1}{K_a} = \frac{1}{K}$$

Hence, error table be;

Scanned with CamScanner

Error (e_{ss}) table

Input	Type – O	Type – I	Type – II
Unit step input	$\frac{1}{1+K}$	0	0
Unit ramp input	∞	$\frac{1}{K}$	0
Unit parabolic input	∞	∞	$\frac{1}{K}$

EXAMPLE 7.1

The open loop transfer function of a servo system with unity feedback is $G(S) = \frac{10}{S(1+0.1S)}$

i) Obtain the steady state error of the system when subjected to an input of: (a) Unit step, (b) St and (c) $\frac{3}{2}t^2$

ii) Evaluate static error constant and steady state error when input of $r(t) = (1+5t)$ is applied.

Solution:

Here,
$$G(S)H(S) = \frac{10}{S(1+0.1S)}$$

Now,
$$K_p = \lim_{S \to 0} G(S)H(S) = \infty$$
$$K_v = \lim_{S \to 0} SG(S)H(S) = 10$$
and, $$K_a = \lim_{S \to 0} S^2G(S)H(S) = 0$$

i) a) For unit step input,
Steady state error $(e_{ss}) = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$

b) For ramp input
$r(t) = (1+5t)$
Steady state error $(e_{ss}) = \frac{5}{K_v} = \frac{5}{10} = \frac{1}{2} = 0.5$

c) For parabolic input;
$r(t) = \frac{3}{2}t^2$
Steady state error $(e_{ss}) = \frac{1}{K_a} \times \frac{3}{2} = \frac{1}{0} = \infty$

ii) Now,
$r(t) = (1+5t)$
The input consists of unit step input and ramp input;
so, Steady state error $(e_{ss}) = \frac{1}{1+K_p} + \frac{5}{K_v} = \frac{1}{1+\infty} + \frac{5}{10} = 0.5$

EXAMPLE 7.2

A unity feedback system has $G(S) = \frac{K(2S+1)}{S(4S+1)(S+1)^2}$. Determine value of K if the steady state value of error has to be less than 0.2 when an input $r(t) = (1+50t)$ is applied.

Solution:

The input r(t) ... ramp input, so we need to calculate only ...

$$K_p = \lim_{S \to 0} G(S)H(S) = \infty$$
$$K_v = \lim_{S \to 0} SG(S)H(S) = K$$

Now,
Steady state error $(e_{ss}) = \frac{1}{1+K_p} + 50\left(\frac{1}{K_v}\right) = \frac{1}{1+\infty} + 50\left(\frac{1}{K}\right) = 0 + \frac{50}{K} = \frac{50}{K}$

From the question; we have,
$e_{ss} < 0.2$
i.e., $\frac{50}{K} < 0.2$
or, $K > \frac{50}{0.2}$
$\therefore K > 250$
i.e., the value of K should be greater than 250 for e_{ss} to be less than 0.2.

EXAMPLE 7.3

Consider a unity feedback system with the closed loop transfer function: $\frac{C(S)}{R(S)} = \frac{(KS+b)}{S^2+aS+b}$

Determine:
i) Open loop transfer function
ii) Steady state error in unit ramp input [2066 Magh, 2069 Chaitra]

Solution:
Given that;
Closed loop transfer function $\frac{C(S)}{R(S)} = \frac{(KS+b)}{S^2+aS+b}$

and, $\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)}$; considering negative feedback system

For unity feedback system;

i) $H(S) = 1$;
$$\frac{G(S)}{1+G(S)} = \frac{(KS+b)}{S^2+aS+b}$$
or, $(S^2+aS+b)G(S) = (KS+b) + (KS+b)G(S)$
or, $(S^2+aS+b-KS-b)G(S) = (KS+b)$
or, $[S^2+S(a-K)]\,G(S) = (KS+b)$
$\therefore$ $\boxed{G(S) = \frac{(KS+b)}{S[S+(a-K)]}}$; which is open loop transfer function.

ii) The input $r(t) = t$ is given; so, we need to calculate only K_v.
$$K_v = \lim_{S \to 0} SG(S)H(S) = \lim_{S \to 0} SG(S) = \lim_{S \to 0} S\left[\frac{(KS+b)}{S(S+a-K)}\right] = \frac{b}{(a-K)}$$

and, the steady state error is;
$$e_{ss} = \frac{1}{K_v} = \frac{a-K}{b}$$

Scanned with CamScanner

$\therefore$ $\boxed{e_{ss} = \frac{a - K}{b}}$ which is steady state error.

EXAMPLE 7.4

Find all static error constant for a unity feedback system with feed forward transfer function $G(S) = \frac{1000}{S(S+10)(S+100)}$. Evaluate steady state error if system is excited with $r(t) = 2 + t$. [2070 Chaitra]

Solution:

Here,

$$G(S)H(S) = \frac{1000}{S(S+10)(S+100)}; H(S) = 1$$

i) Static position error constant (K_p)

$$K_p = \lim_{S \to 0} G(S)H(S) = \infty$$

$\therefore$ $\boxed{K_p = \infty}$

ii) Static velocity error constant (K_v)

$$K_v = \lim_{S \to 0} SG(S)H(S) = \frac{1000}{10 \times 100} = 1$$

$\therefore$ $\boxed{K_v = 1}$

iii) Static acceleration error constant (K_a)

$$K_a = \lim_{S \to 0} S^2G(S)H(S) = 0$$

$\therefore$ $\boxed{K_a = 0}$

iv) The input $r(t) = 2 + t$ consists of step input and ramp input.

Now,

$$\text{Steady state error } (e_{ss}) = \frac{2}{1+K_p} + \frac{1}{K_v} = \frac{2}{1+\infty} + \frac{1}{1} = 0 + 1 = 1$$

$\therefore$ $\boxed{e_{ss} = 1}$ which is the required steady state error.

EXAMPLE 7.5

The open loop transfer function of a unity feedback system is given by $G(S) = \frac{108}{S^2(S+4)(S^2+3S+12)}$

i) Find static error constants

ii) Steady state error of the system when subjected to an input, $r(t) = 2 + 5t$

[2067 Magh, 2074 Ashwin]

Solution:

Here,

$$G(S)H(S) = \frac{108}{S^2(S+4)(S^2+3S+12)}$$

Static position error constant $(K_p) = \lim_{S \to 0} G(S)H(S) = \infty$

Static velocity error constant $(K_v) = \lim_{S \to 0} SG(S)H(S) = \infty$

Static acceleration error constant $(K_a) = \lim_{S \to 0} S^2G(S)H(S) = \frac{108}{4 \times 12} = 2.25$

iv) Here, input $r(t) = 2 + 5t$ consists of step input and ramp input; so, steady state error becomes;

$$e_{ss} = \frac{2}{1+K_p} + \frac{5}{K_v} = \frac{2}{1+\infty} + \frac{5}{\infty} = 0$$

$\therefore$ $e_{ss} = 0$

EXAMPLE 7.6

The open loop transfer function of a unity feedback system is given by $G(S) = \frac{5}{S(S+2)(S^2+2S+8)}$

Find the static error coefficients and steady state error of the system when subjected to an input by $r(t) = 2 + 5t + 2t^2$. [2067 Ashadh]

Solution:

Here,

$$G(S)H(S) = \frac{5}{S(S+2)(S^2+2S+8)}; H(S) = 1$$

i) Static position error constant $(K_p) = \lim_{S \to 0} G(S)H(S) = \infty$

ii) Static velocity error constant $(K_v) = \lim_{S \to 0} SG(S)H(S) = \frac{5}{2 \times 8} = \frac{5}{16}$

iii) Static acceleration error constant $(K_a) = \lim_{S \to 0} S^2G(S)H(S) = 0$

iv) The input $r(t) = 2 + 5t + 2t^2$ consists of stem input, ramp input and parabolic input;

$$\text{Steady state error } (e_{ss}) = \frac{1}{1+K_p} \times 2 + 5 \times \frac{1}{K_v} + 2 \times \frac{1}{K_a} = \frac{2}{1+\infty} + 5 \times \frac{16}{5} + 2 \times \frac{1}{0}$$

$$= 0 + 16 + \infty = \infty$$

7.2 FREQUENCY DOMAIN SPECIFICATION

See the definition part 6.2.1

i) Resonant peck (M_r)
ii) Resonant frequency (ω_r)
iii) Bandwidth (BW)
iv) Cut off frequency (ω_b)
v) Cut off rate

7.2.1 Gain Margin and Phase Margin

- Gain margin and phase margin of Bode plot: See the definition part 6.5
- Gain margin and phase margin of polar plot: See the definition part 6.6.9

7.3. COMPENSATION AND IT'S TYPE

7.3.1 Compensation

Compensation is the process of making performance specifications as desired. Compensation is required to make unstable system to stable and improve existing system to satisfy or meet required specifications. The system performance is given by the time response and frequency response.

In many practical cases, the adjustment of gain alone may not provide sufficient alteration of the system behavior to meet the given specifications. As frequently increasing the gain will improve the steady state behavior but will result in poor transient characteristics (poor stability or even instability).

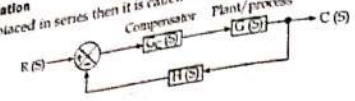
It is therefore necessary to design the system by modifying the structure or by incorporating additional devices (or components) to alter the overall behavior so that the system will behave as desired.

Scanned with CamScanner

7.3.2 Types of Compensation

i) Series compensation

When compensator is placed in series then it is called cascade compensation or series compensation.



Advantage

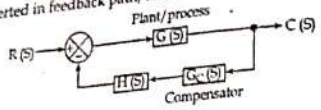
a) It is simple.

Disadvantage

a) It requires additional amplifiers to increase gain.

ii) Feedback compensation or parallel compensation

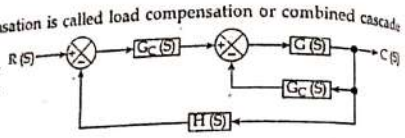
When a compensator is inserted in feedback path, this is called feedback compensation.



iii) Load compensation

A combination of series and feedback compensation is called load compensation or combined cascade and feedback compensation.

The compensator may be electrical, pneumatic or hydraulic type. Mostly electrical networks are used. The simplest networks are:



a) Lead compensator
b) Lag compensator
c) Lead-lag compensator

7.4 DESIGN OF COMPENSATION USING BODE'S PLOT

We used lead compensator and lag compensator using Bode's plot.

In frequency response approach, we specify the transient response performance in an indirect manner *i.e.*, in terms of phase margin gain margin, resonant peak magnitudes (they give rough estimate of the system damping), gain crossover frequency, resonant frequency bandwidth (they give a rough estimate of speed of transient response) and static error constants (they given steady state error (e_{ss}) accurately).

7.4.1 Phase Lead Compensation

The lead compensation using Bode plot can be done with phase lead compensator (*i.e.*, high pass filter or PD compensator)

In lead compensator, zero is closer than pole to origin.

The design procedure for phase lead is;

Step 1

Set static error criteria, determine K, and plot Bode magnitude and phase plot for uncompensated system.

Step 2

From Bode plot; determine phase margin (*PM*).

Let, ϕ = Phase margin of uncompensated system.

ϕ_s = Desired phase margin

ϵ = Margin of safety (5 to 12°; usually 5°)

Then,

$$\phi_m = (\phi_s - \phi + \epsilon)$$

NOTE

If $\phi_m > 60°$, two identical network each contributing maximum lead of $\left(\frac{\phi_m}{2}\right)$ are used.

i.e., we use two compensators, each have $\phi_{m(new)} = \frac{\phi_m}{2}$ (if $\phi_m > 60°$)

Step 3 Calculate attenuation constant (a)

$$a = \frac{1 + \sin\phi_m}{1 - \sin\phi_m}$$

Step 4 From Bode magnitude plot

Calculate ω_m at which uncompensated system will have gain $-10\log_{10} a$

i) Determine: $-10\log_{10} a$

ii) Draw a horizontal line from $-10\log_{10} a$ in Bode magnitude plot (dotted line).

iii) Draw a vertical line from crossing point of horizontal dotted line with magnitude plot. This gives frequency ω_m.

Step 5

Determine T, using ω_m and a

$$T = \frac{1}{\omega_m\sqrt{a}}$$

Step 6

Find transfer function of phase lead network (compensator)

i.e., $G_c(S) = \frac{1 + aTS}{1 + TS}$

Step 7

Determine the overall transfer function for a compensated system

$$G'(S) = G_c(S) \times G(S)$$

Step 8

Draw a new Bode plot for G'(S) and check whether it meets specified criteria or not

Step 9

If not; continue the process from step 2, with higher value of ϵ (margin of safety)

EXAMPLE 7.7

Design a suitable lead compensator for a system having open loop transfer function $G(S) = \frac{K}{S(1+0.1S)(1+0.001S)}$ Such that the compensated system should have phase margin (PM) of at least 45° and static velocity error constant at least 1000. [2068 Chaitra, 2074 Ashwin]

Solution:

Step 1 Set static error criteria

i.e., $K_v \geq 1000$

Now,

$$K_v = \lim_{S \to 0} SG(S)H(S)$$

or, $$1000 = \lim_{S \to 0} S \times \frac{K}{S(1+0.1S)(1+0.001S)}$$

Scanned with CamScanner

$$\text{or,}\quad 1000 = \lim_{S \to 0} \frac{K}{S(1+0.1S)(1+0.001S)} = \frac{K}{1}$$

$$\therefore\quad K = 1000$$

Step 2 Sketch Bode magnitude and phase plot

i) Put, $S = j\omega$

$$G(j\omega) = \frac{1000}{j\omega(1+0.1j\omega)(1+0.001j\omega)}$$

ii) Determining corner frequencies

Corner frequency for poles; $\omega_1 = \frac{1}{0.1} = 10$ rad/sec. (smallest)

$$\omega_2 = \frac{1}{0.001} = 1000 \text{ rad/sec.}$$

Now,

Starting frequency $(\omega) = \frac{1}{10} \times \text{Lowest} = \frac{1}{10} \times 10 = 1$ rad/sec.

iii) Magnitude plot

a) Starting magnitude $M = 20\log_{10} K - 20N\log_{10}\omega$

Here,

$N = 1$ = Type I system

$K = 1000$

and, $\omega = 1$ rad/sec.

so, $M = 20\log_{10}(1000) - 20N\log_{10}(1) = 60 - 0 = 60$ dB

b) From $\omega = 1$ to $\omega = 10$ rad/sec.; Slope = -20 dB/dec.

From $\omega = 10$ to $\omega = 1000$ rad/sec.; Slope = $-20 + (-20) = -40$ dB/dec.

From $\omega = 1000$ to $\omega =$ higher; Slope = $-40 + (-20) = -60$ dB/dec.

iv) Phase plot

$$\phi_{Total} = \angle G(j\omega) = -90° - \tan^{-1}(0.1\omega) - \tan^{-1}(0.001\omega)$$

ω	ϕ_{Total}	ω	ϕ_{Total}
1	− 95.76°	150	− 184.71°
5	− 116.85°	200	− 188.44°
10	− 135.57°	500	− 205.41°
50	− 171.55°	1000	− 224.43°
100	− 180°	10000	− 264.23°

Plotting magnitude and phase plot as an uncompensated system

Step 3

We get, phase margin from Bode plot;

$PM = \phi = 0°$

so, ϕ_s = Desired PM = 45°

ϵ = Margin of safety = 5°

Now,

$\phi_m = \phi_s - \phi + \epsilon = 45° - 0° + 5° = 50°$ (and $\phi_m < 60°$)

Since, $\phi_m < 60°$; we need only one compensator.

Step 4 Attenuation constant

$$\text{Attenuation constant (a)} = \frac{1+\sin\phi_m}{1-\sin\phi_m} = \frac{1+\sin 50°}{1-\sin 50°} = 7.5486$$

Step 5 Calculate (ω_m)

We have,

i) Magnitude of $-10\log_{10} a = -10\log_{10}(7.54) = -8.779$ dB

ii) Draw horizontal dotted line from − 8.78 dB to the intersection with magnitude plot.

iii) Draw vertical line from that intersection point, we get value of ω_m.

$\therefore\quad \omega_m = 170$ rad/sec. (from graph)

Step 6

$$T = \frac{1}{\omega_m\sqrt{a}} = \frac{1}{170\sqrt{7.54}} = 2.141 \times 10^{-3} \text{ sec.}$$

Step 7

Transfer function of compensator is given by;

$$G_c(S) = \frac{1+aTS}{1+TS} = \frac{1+7.54\times 2.141\times 10^{-3}S}{1+2.141\times 10^{-3}S} = \frac{1+0.01614S}{1+0.00214S}$$

Step 8 Overall transfer function

$$G'(S) = G_c(S) \times G(S) = \frac{1000(1+0.01614S)}{S(1+0.1S)(1+0.001S)(1+0.00214S)}$$

Step 9 Drawing Bode plot

i) Corner frequency; $\omega_1 = \frac{1}{0.1} = 10$ (pole) (lowest)

$$\omega_2 = \frac{1}{0.001} = 1000 \text{ (pole)}$$

$$\omega_3 = \frac{1}{0.00214} = 467.29 \text{ (pole)}$$

$$\omega_4 = \frac{1}{0.01614} = 61.95 \text{ (zero)}$$

ii) Starting frequency $(\omega) = \frac{1}{10} \times 10 = 1$ rad/sec.

iii) Starting magnitude (M) $= 20\log_{10} K - 20N\log_{10}\omega = 20\log_{10}(1000) - 20\times 1\times\log_{10}(1) = 60$ dB

iv) From $\omega = 1$ to $\omega = 10$ rad/sec.; Slope = -20 dB/dec.

From $\omega = 10$ to $\omega = 61.95$ rad/sec.; Slope = $-20 + (-20) = -40$ dB/dec.

From $\omega = 61.95$ to $\omega = 467.289$ rad/sec.; Slope = $-40 + 20 = -20$ dB/dec.

From $\omega = 467.289$ to $\omega = 1000$ rad/sec.; Slope = $-20 + (-20) = -40$ dB/dec.

From $\omega = 1000$ to $\omega =$ higher; Slope = $-40 + (-20) = -60$ dB/dec.

v) **Phase plot**

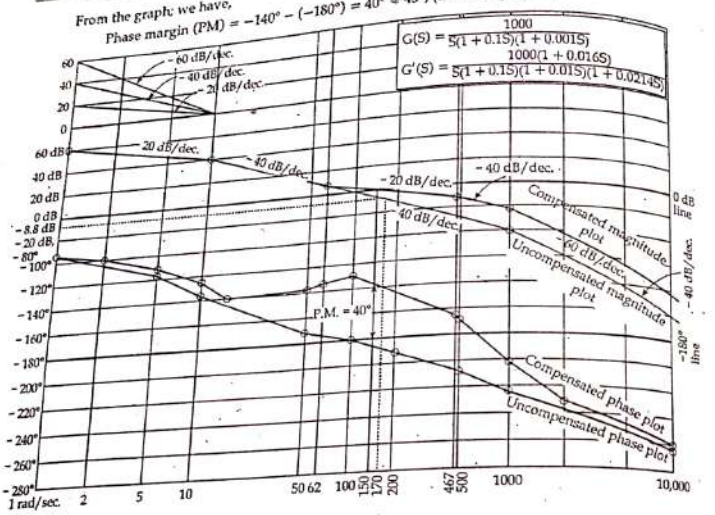
$$\phi_{Total} = \angle G'(j\omega) = -90° - \tan^{-1}(0.1\omega) - \tan^{-1}(0.001\omega) - \tan^{-1}(0.00214\omega) + \tan^{-1}(0.01614$$

ω	ϕ_{Total}	ω	ϕ_{Total}
1	− 94.96°	100	− 133.86°
2	− 99.82°	467.289	− 166.37°
5	− 112.85°	500	− 169.42°
10	− 127.63°	1000	− 202.92°
20	− 139.14°	2000	− 231.77°
50	− 138.75°	10000	− 261.91°
61.95	− 136.93°		

Scanned with CamScanner

From the graph; we have,

Phase margin (PM) = $-140° - (-180°) = 40° \approx 45°$; (almost equal)



EXAMPLE 7.8

Design a lead compensator for a system having open loop transfer function $G(S) = \frac{K}{S(1+0.2S)}$ with unity feedback system such that $K_v = 10$ and phase margin should be 50°.

[2066 Bhadra, 2066 Magh, 2067 Ashadh]

Solution:

Step 1 Set static error criteria

$$K_v = \lim_{S \to 0} SG(S)H(S)$$

or, $10 = \lim_{S \to 0} S\frac{K}{S(1+0.2S)}$

or, $10 = \frac{K}{1}$

$\therefore K = 10$

Step 2

Draw Bode plot for uncompensated system

$$G(S) = \frac{K}{S(1+0.2S)}$$

Put, $S = j\omega$; then,

$$G(j\omega) = \frac{K}{j\omega(1+0.2j\omega)}$$

i) Corner frequencies $(\omega_1) = \frac{1}{0.2} = 5$ rad/sec.

ii) Starting frequency $(\omega) = \frac{1}{10} \times 5 = 0.1$ (approximate to nearest decade)

iii) Starting magnitude $= 20\log_{10} K - 20N\log_{10}\omega$

iv) and, $K = 10$

$N = 1$ = Type 1 system

and, $\omega = 0.1$

so, $M = 20\log_{10}10 - 20N\log_{10}(0.1) = 20 + 20 = 40$ dB

From $\omega = 0.1$ to $\omega = 5$ rad/sec.; slope $= -20$ dB/dec.

v) From $\omega = 5$ to $\omega =$ higher; slope $= -20 + (-20) = -40$ dB/dec.

vi) Phase angle $(\phi_{Total}) = \angle G(j\omega) = -90° - \tan^{-1}(0.2\omega)$

ω	ϕ_{Total}	ω	ϕ_{Total}
0.1	− 91.14°	10	− 153.43°
0.2	− 92.3°	20	− 165.96°
0.5	− 95.71°	50	− 174.29°
1	− 101.31°	100	− 177.13°
2	− 111.8°	500	− 179.43°
5	− 135°	1000	− 179.71°

Plotting magnitude and phase plot; we get, phase margin.

Step 3

$\phi_{obtained} = 35°$

$\phi_{designed} = \phi_s$ = Desired P.M. = 50°

ϵ = Margin of safety = 5°

so, $\phi_m = \phi_s - \phi + \epsilon = 50° - 5° + 35° = 20°$ ($\phi_m < 60°$)

Hence, only one compensator is used.

Step 4 Calculate attenuation factor

$$\text{Attenuation factor (a)} = \frac{1+\sin\phi_m}{1-\sin\phi_m} = \frac{1+\sin 20°}{1-\sin 20°} = 2.0396$$

Step 5

Calculate gain $= -10\log_{10} a = -10\log_{10}(2.0396) = -3.0955$ dB

From this gain, corresponding frequency is calculated;

so, $\omega_m = 8$ rad/sec.

Step 6 Calculate T

$$T = \frac{1}{\omega_m\sqrt{a}} = \frac{1}{8 \times \sqrt{2.0396}} = 0.087526$$

Step 7

Transfer function of compensator is given by;

$$G_c(S) = \frac{1+aTS}{1+TS} = \frac{1+2.0396 \times 0.087526S}{1+0.0875S} = \frac{1+0.1785S}{1+0.0875S}$$

Step 8

Overall transfer function is given by;

Scanned with CamScanner

$$G'(S) = G_c(S) \times G(S) = \frac{10(1 + 0.1785\,S)}{S(1 + 0.2\,S)(1 + 0.0875\,S)}$$

Step 3

Draw Bode magnitude plot and phase plot for compensated system.

i) Put, $S = j\omega$

$$G'(j\omega) = \frac{10(1 + 0.1785\,j\omega)}{j\omega(1 + 0.2\,j\omega)(1 + 0.0875\,j\omega)}$$

ii) Corner frequencies; $\omega_1 = \frac{1}{0.2} = 5$ rad/sec. (smallest) (pole)

$\omega_2 = \frac{1}{0.0875} = 11.42$ rad/sec. (pole)

$\omega_3 = \frac{1}{0.1785} = 5.60$ rad/sec. (zero)

iii) Starting frequency $(\omega) = \frac{1}{10} \times$ Lowest $= 0.5 \approx 0.1$ (approximation to nearest decade)

iv) Starting magnitude $(M) = 20\log_{10} K - 20N\log_{10}\omega = 20\log_{10} 10 - 20 \times 1 \times \log_{10}(0.1) = 40$ dB

v) From $\omega = 0.1$ to $\omega = 5$ rad/sec.; slope $= -20$ rad/sec.

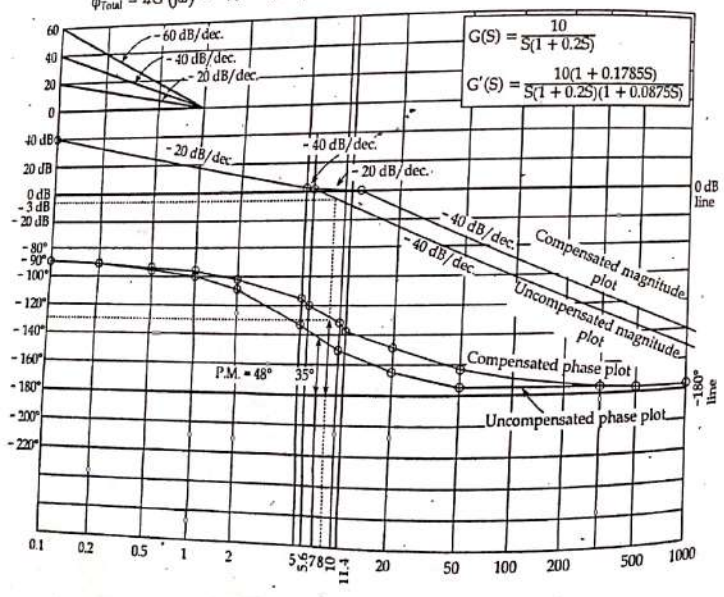
From $\omega = 5$ to $\omega = 5.6$ rad/sec.; slope $= -20 + (-20) = -40$ dB/dec.

From $\omega = 5.6$ to $\omega = 11.42$ rad/sec.; slope $= -40 + (20) = -20$ dB/dec.

From $\omega = 11.42$ to $\omega =$ higher; slope $= -20 + (-20) = -40$ dB/dec.

vi) Phase angle plot,

$$\phi_{Total} = \angle G'(j\omega) = -90° - \tan^{-1}(0.2\omega) - \tan^{-1}(0.0875\omega) + \tan^{-1}(0.1785\omega)$$



ω	ϕ_{Total}	ω	ϕ_{Total}	ω	ϕ_{Total}
0.1	− 90.62°	5	− 116.88°	50	− 167.8°
0.2	− 91.24°	5.6	− 119.35°	100	− 173.82°
0.5	− 93.11°	10	− 133.88°	200	− 177°
1	− 96.19°	11.42	− 137.46°	500	− 179°
2	− 102.08°	20	− 151.87°	1000	≈ − 180°

From the graph; we have,

P.M. $= -132° - (-180°) = 48° \approx 50°$ (almost equal)

EXAMPLE 7.9

The open loop transfer function of type two systems with unity feedback system is given by $G(S) = \frac{K}{S^2(1 + 0.25\,S)}$. Design a lead compensator to meet the following specifications. [2066 Jestha]

i) Acceleration constant $(K_a) = 10$ ii) PM = 35°

Solution:

Step 1 Draw Bode plot for uncompensated system

i) Put, $S = j\omega$ on $G(S)$

$$G(j\omega) = \frac{K}{(j\omega)^2(1 + 0.25\,j\omega)}$$

ii) Corner frequency $(\omega_1) = \frac{1}{0.25} = 4$ rad/sec.

iii) Starting frequency $(\omega) = \frac{1}{10} \times 4 = 0.4$ rad/sec. $\simeq 0.1$ rad/sec. (approximating to nearest decade)

iv) N = 2 = type II system

and, $K_a = \lim_{S \to 0} S^2 G(S)H(S) = \lim_{S \to 0}\left[S^2 \frac{K}{S^2(1 + 0.25\,S)}\right] = \frac{K}{1} = K$

$\therefore \quad K_a = K = 10$

v) Starting magnitude $(M) = 20\log_{10} K - 20N\log_{10}\omega = 20\log_{10} 10 - 20 \times 2 \times \log_{10}(0.1) = 60$ dB

vi) From $\omega = 0.1$ to $\omega = 4$ rad/sec.; slope $= -20 \times 2 = -40$ dB/dec.

From $\omega = 4$ to $\omega =$ higher; slope $= -40 + (-20) = -60$ dB/dec.

vii) **Phase plot**

$$\phi_{Total} = \angle G'(j\omega) = -180° - \tan^{-1}(0.25\omega)$$

ω	ϕ_{Total}	ω	ϕ_{Total}
0.1	− 181.43°	20	− 258.69°
0.2	− 182.86°	50	− 265.42°
0.5	− 187.12°	100	− 267.70°
1	− 194.03°	200	− 268.85°
2	− 206.56°	500	− 269.54°
5	− 231.34°	1000	− 269.77°
10	− 248.19°		

Step 2

From Bode plot; we have,

Scanned with CamScanner

Phase margin (P.M.) $= \phi = -38°$

Desired P.M $= \phi_s = 35°$

Margin of safety $(\epsilon) = 5°$

$\phi_m = \phi_s - \phi + \epsilon = 35° - (-38°) + 5° = 78° (> 60°)$

Since, $\phi_m > 60°$, so, we need two compensators having $\phi'_m = \frac{\phi_m}{2} = \frac{78°}{2} = 39°$ phase margin each

Step 3 Attenuation factor

Attenuation factor $(a) = \frac{1+\sin\phi'_m}{1-\sin\phi'_m} = \frac{1+\sin 39°}{1-\sin 39°} = 4.3955$

Step 4 Zero frequency attenuation

Zero frequency attenuation $= 2 \times (-10\log_{10} a) = 2 \times (-10\log_{10} 4.3955) = -12.86$ dB

Step 5

Calculate frequency ω_m from -12.86 dB line.

From the graph; we have,

$\omega_m = 5.2$ rad/sec.

Step 6 Calculate T

$$T = \frac{1}{\omega_m\sqrt{a}} = \frac{1}{5.2 \times \sqrt{4.3955}} = 0.09173$$

Step 7

Transfer function of compensator $= \frac{1+aTS}{1+TS}$

Since, we need two compensators;

$$G_c(S) = \left(\frac{1+aTS}{1+TS}\right)^2 = \left(\frac{1+4.3955 \times 0.09173\,S}{1+0.09173\,S}\right)^2 = \left(\frac{1+0.4032\,S}{1+0.09173\,S}\right)^2$$

Step 8

Overall transfer function of compensated system is given by;

$$G'(S) = G_c(S) \times G(S) = \frac{10(1+0.4032\,S)^2}{S(1+0.25\,S)(1+0.1135\,S)^2}$$

Step 9 Draw Bode plot again for compensated system G'(S)

i) Put, $S = j\omega$

$$G'(j\omega) = \frac{10(1+0.4032\,j\omega)^2}{(j\omega)^2(1+j0.25\,\omega)(1+j0.1135\,\omega)^2}$$

i) Corner frequencies; $\omega_1 = \frac{1}{0.25} = 4$ rad/sec. (pole)

$\omega_2 = \frac{1}{0.4032} = 2.48$ rad/sec. (smallest) (zero)

$\omega_3 = \frac{1}{0.1135} = 8.81$ rad/sec. (pole)

ii) Starting frequency $(\omega) = \frac{1}{10} \times 2.48 = 0.248 \approx 0.1$ rad/sec. (approximating to nearest decade)

v) Staring magnitude $(M) = 20\log_{10} K - 20N\log_{10}\omega = 20\log_{10} 10 - 20 \times 2 \times \log_{10}(0.1) = 60$ dB

) From $\omega = 0.1$ to $\omega = 2.48$ rad/sec.; slope $= -20 \times 2 = -40$ dB/dec.

From $\omega = 2.48$ to $\omega = 4$ rad/sec.; slope $= -40 + 20 \times 2 = -40 + 40 = 0$ dB/dec.

From $\omega = 4$ to $\omega = 8.81$ rad/sec.; slope $= 0 + (-20) = -20$ dB/dec.

From $\omega = 8.81$ to $\omega =$ higher; slope $= -20 + (-20) \times 2 = -60$ dB/dec.

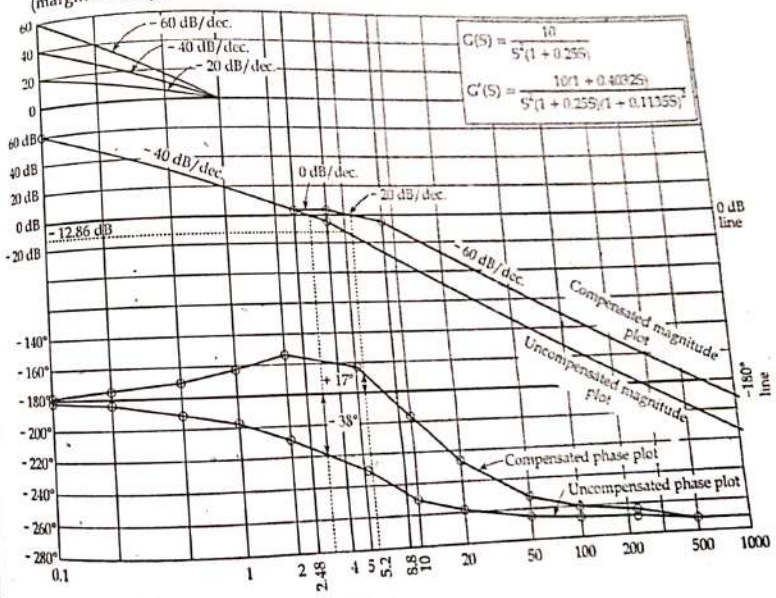
$\Phi_{Total} = \angle G(j\omega) = -180° + 2\tan^{-1}(0.4032\omega) - \tan^{-1}(0.25\omega) - 2\tan^{-1}(0.1135\omega)$

ω	Φ_{Total}	ω	Φ_{Total}
0.1	− 178.12°	20	− 225.28°
0.2	− 176.25°	50	− 251.11°
0.5	− 170.83°	100	− 260.5°
1	− 163.01°	200	− 265.23°
2	− 154.38°	500	− 268.09°
5	− 163.26°	1000	− 269.05°
10	− 193.30°		

From the plot; we have,

P.M. $= -163° - (-180°) = +17°$ (instead of 35°)

The difference in phase margin is due to excessive lag of the fixed part of the system at new crossover frequency. If P.M. = 17° is not acceptable, redesign the system with higher value of ϵ (margin of safety).



7.4.2 Phase Lag Compensation (PI Controller)

In phase lag network, the compensator have pole nearer to origin then zero.

The design procedure for phase lag network is;

Step 1

Set steady state error requirement and determine the value of K. Draw Bode plot.

Scanned with CamScanner

Step 2

Perform, $\phi = \phi_s + \epsilon$

where, ϕ_s is the specified P.M. = Desired P.M.

ϵ is the margin of safety (5° to 12°).

Step 3

Determine the frequency corresponding to the required phase margin from the phase curve. This frequency is new gain crossover frequency.

Step 4

The magnitude curve is brought to 0 dB line at new gain crossover frequency; where, PM is satisfied. The phase lag network must provide the amount of attenuation equal to value of magnitude curve at new g.c.f.

i.e. $|G(j\omega)|_{\text{new g.c.f}} = -20\log_{10} a$

Step 5 Calculation of T

Now,

$$\frac{1}{aT} = \frac{\text{New g.c.f.}}{10}$$

or, $$T = \frac{10}{a \times \text{New g.c.f.}}$$

Step 6

The transfer function of phase lag compensator is:

$$G_c(S) = \frac{1 + aTS}{1 + TS}$$

Step 7

Overall transfer function for the compensated system:

$$G'(S) = G_c(S) \times G(S)$$

Step 8

Draw Bode plot and check whether the requirement is fulfilled or not. If not, repeat the process from step 2 with higher value of ϵ (margin of safety).

EXAMPLE 7.10

Design a compensator which can decrease the band width of given system $G(S) = \frac{K}{S(S+2)(S+20)}$ to meet the following specifications: $K_v \geq 20\ \text{sec}^{-1}$ and P.M. $\geq 35°$ [2070 Ashadh]

Solution:

Step 1

$K_v \geq 20\ \text{sec}^{-1}$

$$K_v = \lim_{S \to 0} S\,G(S)H(S)$$

or, $$20 = \lim_{S \to 0} S\left[\frac{K}{S(S+2)(S+20)}\right]$$

or, $$20 = \frac{K}{2 \times 20}$$

$\therefore$ $K = 800$

Step 2 Draw Bode plot for uncompensated system

i) Put $S = j\omega$

$$G(S) = \frac{800}{S(S+2)(S+20)} = \frac{800}{2 \times 20 \times S\left(1+\frac{S}{2}\right)\left(1+\frac{S}{20}\right)} = \frac{20}{S(1+0.5S)(1+0.05S)}$$

$$= \frac{20}{j\omega(1+j0.5\,\omega)(1+j0.05\,\omega)}$$

ii) Corner frequencies; $\omega_1 = \frac{1}{0.5} = 2$ rad/sec. (pole)

$$\omega_2 = \frac{1}{0.05} = 20 \text{ rad/sec. (pole)}$$

iii) Starting corner frequency (ω) $= \frac{1}{10} \times \text{Lowest} = \frac{1}{10} \times 2$

$= 0.2$ rad/sec.

≈ 0.1 rad/sec. (approximating to nearest decade scale)

iv) We have,

N = Type of system = 1

$\omega = 0.1$

and, K = 20

so, Starting magnitude $= 20\log_{10} K - 20N\log_{10}\omega = 20\log_{10}20 - 20 \times 1 \times \log_{10}(0.1) = 46.02$ dB

v) **Slope**

From $\omega = 0.1$ to $\omega = 2$ rad/sec.; slope = −20 dB/dec.

From $\omega = 2$ to $\omega = 20$ rad/sec.; slope = −20 + (−20) = −40 dB/dec.

From $\omega = 20$ to $\omega =$ higher; slope = −40 + (−20) = −60 dB/dec.

vi) **Phase angle**

$$\phi_{\text{Total}} = \angle G'(j\omega) = -90° - \tan^{-1}(0.5\omega) - \tan^{-1}(0.05\omega)$$

ω	ϕ_{Total}	ω	ϕ_{Total}
0.1	− 93.15°	20	− 219.29°
0.2	− 96.28°	50	− 245.90°
0.5	− 105.47°	100	− 257.54°
1	− 119.43°	200	− 263.71°
2	− 140.71°	500	− 267.48°
5	− 172.23°	1000	− 268.74°
10	− 195.25°		

Step 3

$\phi = \phi_s + \text{Marginal safety} = \phi_s + \epsilon = 35° + 5° = 40°$

Step 4

Phase margin = 40°

or, $180° + \phi = 40°$

or, $\phi = -140°$

i) Draw horizontal line (dotted) from − 140° angle.

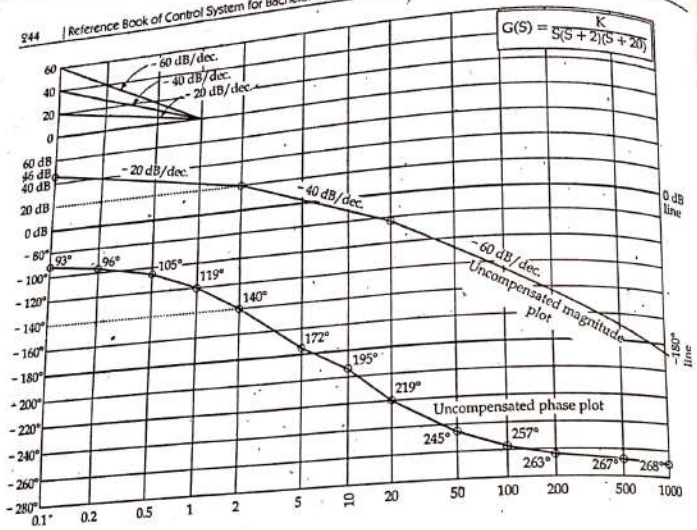
ii) Draw a dotted vertical line from intersection of that horizontal line with phase plot.

iii) Obtain the value of ω_m at that point.

From the graph, we get,

New g.c.f. $= \omega_m = 2$ rad/sec.

Scanned with CamScanner



Step 5

$|G(j\omega)|_{\omega_m} = -20 \log_{10} a$

i) The vertical line (from $\omega_m = 2$ rad/sec.) intersects the magnitude plot as well.

ii) The point of intersection corresponds of + 20 dB.

$20 = -20 \log_{10} a$

or, $-\left(\frac{20}{20}\right) = \log_{10} a$

or, $-1 = \log_{10} a$

Taking antilog; we have,

$10^{-1} = a$

$\therefore$ Attenuation constant (a) = 0.1

Step 6 Calculation of T

$$T = \frac{10}{a \times \text{New g.c.f.}} = \frac{10}{0.1 \times 2} = 50$$

Step 7

Transfer function of compensator is;

$$G_c(S) = \frac{1 + aTS}{1 + TS} = \frac{1 + 0.1 \times 50S}{1 + 50S} = \frac{1 + 5S}{1 + 50S}$$

Step 8

Hence, the overall transfer function of compensated system is;

$$G'(S) = G_c(S) \times G(S) = \frac{20(1 + 5S)}{S(1 + 50S)(1 + 0.5S)(1 + 0.05S)}$$

Step 9 Draw Bode plot for compensated system G'(S)

Put, $S = j\omega$

i) $$G'(j\omega) = \frac{20(1 + j5\,\omega)}{j\omega(1 + j50\,\omega)(1 + j0.5\,\omega)(1 + j0.05\,\omega)}$$

ii) Corner frequencies; $\omega_1 = \frac{1}{0.5} = 2$ rad/sec. (pole)

$\omega_2 = \frac{1}{0.05} = 20$ rad/sec. (pole)

$\omega_3 = \frac{1}{50} = 0.02$ rad/sec. (smallest) (pole)

$\omega_4 = \frac{1}{5} = 0.2$ rad/sec. (zero)

iii) Starting frequency $= \frac{1}{10} \times 0.02$

or, $\omega = 0.002$ rad/sec.

or, $\omega = 0.001$ rad/sec. (rounding)

Approximating to nearest decade scale; take $\omega = 0.01$ rad/sec.

$\therefore$ $\omega = 0.01$ rad/sec.

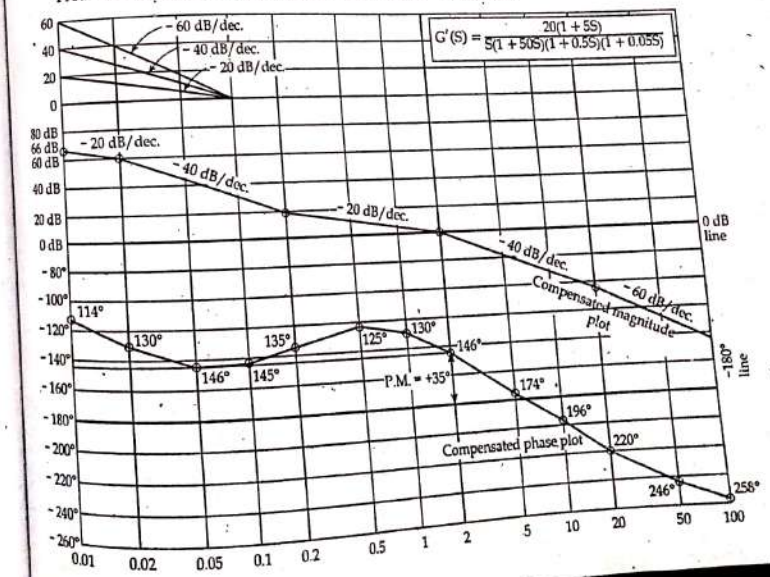
Here, we don't take $\omega = 0.001$ as starting frequency. Because on doing so, the corner frequency does not fit in Bode plot. So; we take, $\omega = 0.01$ rad/sec..

iv) **Slope**

From $\omega = 0.01$ to $\omega = 0.02$ rad/sec.; slope = −20 dB/dec.

From $\omega = 0.02$ to $\omega = 0.2$ rad/sec.; slope = −20 + (−20) = −40 dB/dec.

From $\omega = 0.2$ to $\omega = 2$ rad/sec.; slope = −40 + (20) = −20 dB/dec.



Scanned with CamScanner

From $\omega = 2$ to $\omega = 20$ rad/sec.; slope $= -20 + (-20) = -40$ dB/dec.

From $\omega = 20$ to $\omega =$ higher; slope $= -40 + (-20) = -60$ dB/dec.

v) Starting magnitude $(M) = 20\log_{10} K - 20N\log_{10}\omega = 20\log_{10} 20 - 20 \times 1 \times \log_{10}(0.01) = 66.02$ dB

vi) Phase plot

$\phi_{Total} = \angle G'(j\omega) = -90° - \tan^{-1}(0.5\omega) - \tan^{-1}(0.05\omega) - \tan^{-1}(50\omega) + \tan^{-1}(5\omega)$

ω	ϕ_{Total}	ω	ϕ_{Total}
0.01	− 114°	2	− 146°
0.02	− 130°	5	− 174°
0.05	− 146°	10	− 196°
0.1	− 145°	20	− 220°
0.2	− 135°	50	− 246°
0.5	− 125°	100	− 258°
1	− 130°		

7.5. DESIGN OF COMPENSATION USING ROOT LOCUS

7.5.1 Design Step for Lead Compensator

In lead compensator, zero is closer than pole to origin.

Step 1

Draw root locus of uncompensated system

Step 2

From the transient response specifications (ζ, ω_n). Determine the desired location of dominant close loop poles by applying $S = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$.

Step 3

Collect all the open loop poles and zeros to one of the dominant closed loop pole (either upward or downward) and measure the angles, then find out the sum of angles and calculate the angle deficiency ϕ to be added by a lead compensator

i.e., $\phi + [\angle G(S)$ at dominant pole$] = -180°$

Step 4

Calculate the point at which poles and zeros lies and determine the value of α and T.

Pole at $S = -\frac{1}{\alpha T}$

Zero at $S = -\frac{1}{T}$

Step 5

The transfer function of compensator is;

$$G_c(S) = \frac{K_c\left(S+\frac{1}{T}\right)}{\left(S+\frac{1}{\alpha T}\right)}$$

and, $\alpha < 1$

Step 6

The overall transfer function for a compensated system is;

$G'(S) = G_c(S) \times G(S)$

Step 7

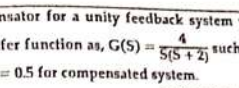
Determine the open loop gain (K) of a compensated system by magnitude criteria.

Step 8

Check whether all the performance specifications have been met or not, and draw root locus.

EXAMPLE 7.10

Design a lead compensator for a unity feedback system with its feed forward transfer function as, $G(S) = \frac{4}{S(S+2)}$ such that $\omega_n = 4$ rad/sec. and $\zeta = 0.5$ for compensated system.



[2068 Baishakh, 2071 Chaitra, 2072 Chaitra, 2073 Chaitra]

Solution:

Step 1 Draw root locus

i) Here,

$$G(S) = \frac{4}{S(S+2)}$$

Open loop poles at $S_1 = 0$

Open loop poles at $S_2 = -2$

Plot these on graph.

ii) Root locus lies in between (0 to − 2)

iii) Number of open loop poles (P) = 2

Number of open loop zeros (Z) = 0

Number of root loci (N) = P = 2; since, P > Z

iv) Number of asymptote = P − Z = 2 − 0 = 2

Angle of asymptote $(\phi) = \frac{2m+1}{P-Z} \times 180°$

so, $\phi_1 = \frac{2\times 0+1}{2} \times 180° = 90°$ at m = 0

$\phi_2 = \frac{2\times 1+1}{2} \times 180° = 270°$ at m = 1

v) Centroid of asymptote $(\sigma_A) = \frac{\sum P - \sum Z}{P-Z} = \frac{(0-2)-0}{2} = -1$

vi) **Breakaway point**

Characteristics equation is;

$1 + G(S)H(S) = 0$

or, $1 + \frac{4}{S(S+2)} = 0$

or, $S^2 + 2S + 4 = 0$

and, we have to calculate $\frac{dK}{dS} = 0$;

For root locus; we need gain,

so, $G(S) = \frac{4K}{S(S+2)}$

so, characteristics equation is;

$1 + G(S)H(S) = 0$

or, $1 + \frac{4K}{S(S+2)} = 0$

Scanned with CamScanner

or, $S^2 + 2S + 4K = 0$

or, $K = -\left(\frac{S^2 + 2S}{4}\right)$

Perform

$$\frac{dK}{dS} = -\left(\frac{2S+2}{4}\right) = -\left(\frac{S+1}{2}\right) = 0$$

or, $-\left(\frac{S+1}{2}\right) = 0$

or, $S + 1 = 0$

or, $S = -1$, which is the breakaway point.

vii) **jω crossing**

The characteristics equation is;

$S^2 + 2S + 4K = 0$

Performing R-H criteria:

S^2	1	4K
S^1	2	
S^0	4K	

The S^0 row can't be zero. Hence, any row can't be zero. So, we concluded that the root locus doesn't intersect imaginary axis.

Step 2 Calculate dominant pole

$$S_d = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -(0.5) \times 4 \pm j \times 4\sqrt{1-(0.5)^2} = -2 \pm j2\sqrt{3} = -2 \pm j3.464$$

Step 3

ϕ + Total $\angle G(S)H(S)$ at dominant pole = $-180°$

or, $$\phi + \left.\frac{4}{S(S+2)}\right|_{S_d=-2\pm j2\sqrt{3}} = -180°$$

or, $$\phi + \frac{0°}{120° \times 90°} = -180°$$

or, $\phi + (-120° - 90°) = -180°$

$\therefore \quad \phi = 30°$

Step 4

From the graph, for upper dominant pole,

i) Join A and origin O.

ii) Draw horizontal line AP.

iii) Bisect $\angle PAO$.

iv) AB is a bisector of $\angle PAO$.

v) Draw angle $= \frac{\phi}{2} = \frac{30°}{2} = 15°$ on both sides of bisector.

vi) $\angle DAB = \angle CAB = 15°$

vii) In lead compensator, point nearer to origin is zero. From the graph; we have,

Zero at $S_1 = -2.96$ at point C

Pole at $S_2 = -5.5$ at point D

Step 5

The transfer function of compensator is;

$$G_c(S) = \frac{K_c\left(S + \frac{1}{T}\right)}{\left(S + \frac{1}{\alpha T}\right)}$$

In numerator zero,

$\frac{1}{T} = 2.96$

$T = 0.338$

In denominator pole,

$\frac{1}{\alpha T} = 5.5$

or, $\frac{2.96}{\alpha} = 5.5$

$\alpha = 0.5382$

Transfer function of compensator $G_c(S) = \frac{K_c(S + 2.96)}{(S + 5.5)}$

Step 6

The overall transfer function of a compensated system is,

$$G'(S) = G_c(S) \times G(S) = \frac{4K_c(S+2.96)}{S(S+2)(S+5.5)} = \frac{K'(S+2.96)}{S(S+2)(S+5.5)}$$

Step 7 Determination of gain K'

Here,

$$\left.\frac{K'(S+2.96)}{S(S+2)(S+5.5)}\right|_{S_d=-2+j2\sqrt{3}} = 1$$

so, $$\frac{K' \times 3.5946}{S + 2\sqrt{3} + 4.9244} = 1$$

$\therefore \quad K' = 18.98$

and, $K_c = \frac{K'}{4} = 4.7456$

Step 8

Hence, the overall transfer function of compensated system is;

$$G'(S) = \frac{18.98(S+2.96)}{S(S+2)(S+5.5)}$$

Step 9 Draw root locus

i) Pole at $S_1 = 0, S_2 = -2$ and $S_3 = -5.5$

Zero at $S_4 = -2.96$

ii) Number of poles (P) = 3

Number of zeros (Z) = 1

Number of root loci (N) = 3 (since, P > Z)

iii) Centroid of asymptote $(\sigma_A) = \frac{\sum P - \sum Z}{P - Z} = \frac{(-2-5.5)-(-2.96)}{3-1} = -2.27$

iv) Number of asymptote = P − Z = 2

Angle of asymptote;

$$\phi_1 = \frac{2 \times 0 + 1}{2} \times 180° = 90° \text{ at } m = 0$$

Scanned with CamScanner

$$\phi_2 = \frac{2\times1+1}{2}\times180° = 270° \text{ at } m = 1$$

v) Root locus lies in between (0 to − 2) and (− 2.96 to − 5.5)

vi) **Break away point**

The characteristics equation is;

$$1 + \frac{K'(S+2.96)}{S(S+2)(S+5.5)} = 0$$

or, $(S^2+2S)(S+5.5) + K'(S+2.96) = 0$

or, $K' = \frac{S^3+7.5S^2+11S}{(S+2.96)}$

Performing $\frac{dK'}{dS} = 0$; we get,

$2S^3 + 16.38S^2 + 44.4S + 32.56 = 0$

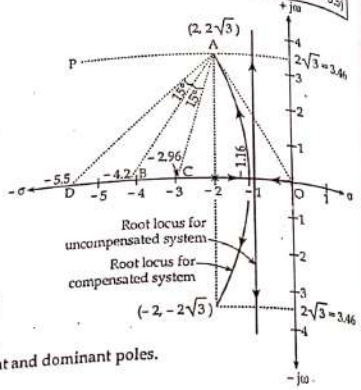
Solving; we get,

$S_1 = -1.16$ (Valid)

$S_2 = -3.516 + j1.3$, $S_3 = -3.516 - j1.3$ } Complex (not valid)

Hence, breakaway point is at $S_1 = -1.16$.

Hence, the root locus passes through this point and dominant poles.



vii) **Characteristics equation**

$S^3 + 7.5S^2 + 11S + (S+2.96)K' = 0$

or, $S^3 + 7.5S^2 + (11+K')S + 2.96K' = 0$

Put, $K' = 18.98$

or, $S^3 + 7.5S^2 + 29.98S + 56.18 = 0$

Solving; we get,

$S_1 = -3.5175$

$S_2 = -2 + j3.46$, $S_3 = -2 - j3.46$ } Dominant poles

Hence, root locus passes through dominant poles.

EXAMPLE 7.11

Design a suitable compensator using root locus for a UCS (uncompensated system) with $G(S) = \frac{10}{S(S+1)(S+4)}$ to meet the specifications. Damping ratio (ζ) = 0.5, Undamped natural frequency = 2 rad/sec. [2072 Kartik]

Solution:

Step 1 Draw root locus

i) Open loop poles at $S_1 = 0$, $S_2 = -1$ and $S_3 = -4$

ii) Root locus lies between (0 to − 1) and (−4 to − ∞)

iii) Centroid of asymptote $(\sigma_A) = \frac{\sum P - \sum Z}{P-Z} = \frac{-5}{3} = -1.667$

iv) Number of asymptote = P − Z = 3

v) Number of root locus (N) = 3 (since, P > Z)

vi) Angle of asymptote; $\phi_1 = \frac{2\times0+1}{3}\times180° = 60°$

$\phi_2 = \frac{2\times1+1}{3}\times180° = 180°$

$\phi_3 = \frac{2\times2+1}{3}\times180° = 300°$

vii) **Breakaway point**

The characteristics equation is given by;

$1 + G(S)H(S) = 0$

or, $1 + \frac{10K}{S(S+1)(S+4)} = 0$

or, $S(S+1)(S+4) + 10K = 0$

or, $10K = -(S^3 + 5S^2 + 4S)$

Here, $\frac{dK}{dS} = 0$ gives;

$(3S^2 + 10S + 4) = 0$

Solving; we get,

$S_1 = -0.465$ (Valid)

$S_2 = -2.868$ (Invalid)

viii) The characteristics equation is given by;

$S^3 + 5S^2 + 4S + 10K = 0$

Solving through R–H array; we get,

S^3	1	4
S^2	5	10K
S^1	$(4-2K)$	
S^0	10	

Again on solving; we get,

$K = 2$ (only zero possible row is S^1)

and, $S = \pm j2$ [by solving auxillary equation $A(S) = 5S^2 + 10K$]

Step 2 Dominant pole

$S_d = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -(0.5)\times2 \pm j\times2\sqrt{1-(0.5)^2} = -1 \pm j\sqrt{3} = -1 \pm j1.73$

Step 3

Angle at upper dominant pole for G(S)H(S) is given by;

$$\angle G(S)H(S) \text{ at dominant pole} = \angle\left[\frac{10}{S(S+1)(S+4)}\right]_{\text{at } S_d=-1+j\sqrt{3}}$$

$$= \frac{0°}{120°\times90°\times30°} = -(120°+90°+30°) = -240°$$

Now,

$\phi - 240° = 180°$

or, $\phi = 60°$

Step 4 Calculate $\left(\frac{\phi}{2}\right) = 30°$

i) Join A and origin O.

ii) Draw horizontal line AP.

Scanned with CamScanner

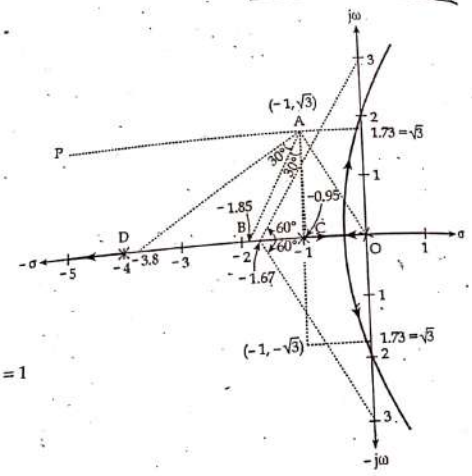
iii) Bisect ∠PAO.
iv) AB is bisector of ∠PAO.
v) Draw angle 30° on each side of bisector.
vi) ∠DAB = ∠CAB = 30°
vii) In lead compensator zero is nearer to origin.
From the graph;
Zero at $S_1 = -0.95$ at point C
Pole at $S_2 = -3.8$ at point D

Step 5
The transfer function of compensator is given by;

$$G_c(S) = \frac{K_C\left(S + \frac{1}{T}\right)}{\left(S + \frac{1}{\alpha T}\right)}$$

$$G(S) = \frac{10}{S(S+2)(S+4)}$$

$$G'(S) = \frac{22.8(S+0.95)}{S(S+1)(S+3.8)(S+4)}$$



and, $\left(\frac{1}{T}\right) = 0.95$ implies $T = 1.0526$

$\left(\frac{1}{\alpha T}\right) = 3.8$ implies $\alpha = 0.25$

$\therefore \quad G_c(S) = \frac{K_C(S+0.95)}{(S+3.8)}$

Step 6 Overall transfer function

$$G'(S) = \frac{4K_C(S+0.95)}{S(S+1)(S+3.8)(S+4)}$$

$$= \frac{K'(S+0.95)}{S(S+1)(S+3.8)(S+4)}$$

Step 7
The value of gain K' can be found by;

$$\left|\frac{K'(S+0.95)}{S(S+1)(S+3.8)(S+4)}\right|_{S_d=-1+j\sqrt{3}} = 1$$

or, $\frac{K' \times 1.732}{S \times \sqrt{3} \times 3.292 \times 2\sqrt{3}} \simeq 1$

or, $K' = 22.80$

and, $K_C = \frac{K'}{10} = 2.280$

Step 8

Drawing root locus for $G'(S) = \frac{22.8(S+0.95)}{S(S+1)(S+3.8)(S+4)}$

The root locus will pass through $S_d = -1 \pm j\sqrt{3}$ for $G'(S)$.

7.5.2 Design Step for Lag Compensator

In lag compensator pole is closer than zero to origin.

Step 1
Draw the root locus of the system.

Step 2
Draw a line at an angle $\theta = \cos^{-1}\zeta$, with negative real axis and locate dominant pole.

Step 3
Determine value of K at an intersection of root locus with this line.

Step 4
Find transfer function of uncompensated system with K.

Step 5
Find $K_{v\,(old)} = \lim_{S \to 0} SG(S)H(S)$; hence,

$\alpha = \frac{K_{v\,(old)}}{K_{v\,(new)}}$; where, $K_{v\,(new)}$ = Desired K_v.

Step 6 Transfer function of compensator

$$G_c(S) = \frac{K_C\left(S + \frac{1}{\alpha T}\right)}{\left(S + \frac{1}{T}\right)}$$

Generally, 10% of distance between origin and the first pole on –ve real axis is taken for the location of compensator zero.

i.e., 10% of that distance $= \frac{1}{\alpha T}$

Step 7
Obtain the value of ω_n

Step 8 Find location of dominant pole

$$S_d = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Step 9
Find value of K_C using magnitude criteria in overall transfer function $G'(S) = G_c(S) \times G(S)$.

Step 10
Check the steady state requirement and find the overall transfer function of compensated system.

EXAMPLE 7.12

Design a lag compensator for a system whose open loop transfer function is $G(S) = \frac{K}{S(S+1)(S+4)}$ to meet the following requirements. $\zeta = 0.5$, $t_s = 10$ sec., $K_v \geq 5$ sec^{-1}

Solution:

Step 1 Sketch root locus
We have already done in lead compensator, check out.

Step 2
Drawing a line with $\zeta = 0.5$
We know that;

$\cos\theta = \zeta$

or, $\theta = \cos^{-1}\zeta = \cos^{-1}(0.5) = 60°$

Plot 60° line from origin on either side of negative real axis. This line intersects the root locus and th
point is dominant pole.

Step 3
The value of K can be found through graphical process.

Scanned with CamScanner

In the graph;

1 unit = 2 cm

or, 1 cm = 0.5 unit

Now, measuring the scale;

Distance from (S = −4) pole to upper dominant pole AC = 7.4 cm = 3.7 unit

Distance from (S = −1) pole to upper dominant pole AB = 1.8 cm = 0.9 unit

Distance from (S = 0) pole to upper dominant pole AO = 1.5 cm = 0.75 unit

Now,

$$K = \frac{\text{Multiplication of distance of every pole to dominant pole}}{\text{Multiplication of distance of every zero to dominant pole}}$$

$$= AC \times AB \times AO = 3.7 \times 0.9 \times 0.75$$

$$= 2.4975 \approx 2.5$$

Step 4 Transfer function of uncompensated system

$$G(S) = \frac{K}{S(S+1)(S+4)} \text{ for } \zeta = 0.5$$

Step 5

$$K_{v\,(new)} = 5\ sec^{-1}$$

$$K_{v\,(old)} = \lim_{S \longrightarrow 0} SG(S)H(S) = \lim_{S \longrightarrow 0} S\left[\frac{2.5}{S(S+1)(S+4)}\right] = \frac{2.5}{4} = 0.625$$

Hence,

$$\alpha = \frac{K_{v\,(old)}}{K_{v\,(new)}} = \frac{0.625}{5} = 0.125$$

Step 6

T.F. of compensator is given by;

$$G_c(S) = \frac{K_C\left(S + \frac{1}{T}\right)}{\left(S + \frac{1}{\alpha T}\right)}$$

Distance between origin to nearest pole (*i.e.*, S = −1) is = 2 cm = 1 unit

This is taken for location of compensator zero.

so, 10% of 1 unit is $\left(\frac{1}{\alpha T}\right)$

$\therefore \quad \frac{1}{\alpha T} = 1 \times 10\% = 0.1$

or, $\frac{1}{T} = 0.1\alpha = 0.1 \times 0.125 = 0.0125$

and, Value of $\left(\frac{1}{\alpha T}\right) = 0.1$

so, $G_c(S) = \frac{K_C(S+0.1)}{(S+0.0125)}$

Hence, the transfer function of compensator is given by;

$$G_c(S) = \frac{K_C(S+0.1)}{(S+0.0125)}$$

Step 7 Find undamped natural frequency (ω_n)

By setting time; we have,

$$t_s = \frac{4}{\zeta\omega_n}$$

or, $10 = \frac{4}{0.5 \times \omega_n}$

or, $\omega_n = 0.8$ rad/sec.

Step 8 Find dominant pole through which new compensated system passes

$$S_d = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$= -(0.5) \times 0.8 \pm j0.8\sqrt{1-(0.5)^2}$$

$$= -0.4 \pm j\frac{2}{5}\sqrt{3} = -0.4 \pm j0.693$$

Step 9 Overall transfer function

$$G'(S) = G_c(S) \times G(S)$$

$$= \frac{K_C(S+0.1)}{(S+0.0125)} \times \left[\frac{2.5}{S(S+1)(S+4)}\right]$$

$$= \frac{2.5K_C(S+0.1)}{S(S+1)(S+4)(S+0.0125)}$$

Now, using magnitude criteria; we have,

$$|G'(S)|_{S_d} = 1$$

or, $\left|\frac{2.5K_C(S+0.1)}{S(S+1)(S+4)(S+0.0125)}\right|_{S_d=-0.4+j0.693} = 1$

or, $\frac{2.5 \times K_C \times 0.755}{0.8 \times 0.9166 \times 3.666 \times 0.794} = 1$

or, $0.8843K_C = 1$

$\therefore \quad K_C = 1.13$

Step 10

Hence, overall transfer function;

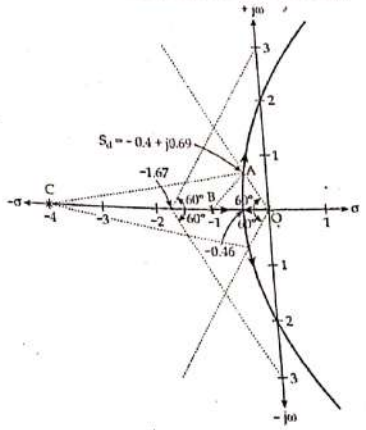
$$\text{T.F.} = G'(S) = G_c(S) \times G(S)$$

$$= \frac{2.5 \times 1.13 \times (S+0.1)}{S(S+1)(S+4)(S+0.0125)}$$

$\therefore \quad \boxed{G'(S) = \frac{2.825(S+0.1)}{S(S+1)(S+4)(S+0.0125)}}$ which is the required transfer function.

$$\boxed{G(S) = \frac{K}{S(S+1)(S+4)}}$$

$$\boxed{G'(S) = \frac{2.825(S+0.1)}{S(S+1)(S+4)(S+0.0125)}}$$



Step 11

Check whether K_v is meet or not.

$$K_v = \lim_{S \longrightarrow 0} SG'(S) = \lim_{S \longrightarrow 0} S\left[\frac{2.825(S+0.1)}{S(S+1)(S+4)(S+0.0125)}\right] = \frac{2.825 \times 0.1}{1 \times 4 \times 0.0125} = 5.65 \approx 5\ sec^{-1}$$

EXAMPLE 7.13

Design a lag compensator whose forward path transfer function of a unity feedback system is given by $G(S) = \frac{K}{S(S+4)(S+5)}$ for $\zeta = 0.707$, $\omega_n = 2$ rad/sec. and $K_v \geq 5\ sec^{-1}$

Solution:

Step 1

Sketch root locus for $G(S) = \frac{K}{S(S+4)(S+5)}$

Scanned with CamScanner

i) Open loop poles at $S_1 = 0, S_2 = -4$ and $S_3 = -5$

ii) Root locus lies between (0 to -4) and (-5 to $-\infty$).

iii) Centroid of asymptote $(\sigma_A) = \frac{\sum P - \sum Z}{P - Z} = \frac{-9-0}{3} = -3$

iv) Number of asymptote $= P - Z = 3 - 0 = 3$

Angle of asymptote; $\phi_1 = \frac{2\times 0+1}{3} \times 180° = 60°$ for $m = 0$

$\phi_2 = \frac{2\times 1+1}{3} \times 180° = 180°$ for $m = 1$

$\phi_3 = \frac{2\times 2+1}{3} \times 180° = 300°$ for $m = 2$

v) Number of root locus $(N) = P = 3$ (since $P > Z$)

vi) **Breakaway point**

The characteristics equation is given by;

$$1 + G(S)H(S) = 0$$

or, $$1 + \frac{K}{S(S+4)(S+5)} = 0$$

or, $$S^3 + 9S^2 + 20S + K = 0$$

Perform $\frac{dK}{dS} = 0$;

$$-(3S^2 + 18S + 20) = 0$$

or, $$3S^2 + 18S + 20 = 0$$

Solving; we get,

$S_1 = -1.47$ (Valid)

$S_2 = -4.52$ (Invalid)

vii) **jω crossing**

The characteristics equation is given by;

$$1 + G(S)H(S) = 0$$

or, $$S^3 + 9S^2 + 20S + K = 0$$

Now, forming R-H array; we have,

S^3	1	20
S^2	9	K
S^1	$\frac{(K-180)}{9}$	
S^0	K	

The only row which can be zero is S^1 row.

so, $$\frac{(K-180)}{9} = 0$$

and, $K = 180$

The auxiliary equation is given by;

$$A(S) = 9S^2 + K = 0$$

or, $$9S^2 + 180 = 0$$

or, $$S^2 = -20$$

$\therefore$ $$S = \pm j2\sqrt{5} = \pm j4.47$$

Hence, the root locus crosses imaginary axis in $\pm j4.47$.

Scanned with CamScanner

Step 2 Drawing a line with $\zeta = 0.707$

We know that;

$$\cos\theta = \zeta$$

or, $$\theta = \cos^{-1}\zeta = \cos^{-1}(0.707) = 45^\circ$$

Plot 45° on each side of negative real axis. The line intersects root locus and gives rise to dominant pole.

Step 3 Value of K can be found through graphical process

In the graph; we have,

$$1 \text{ unit} = 2 \text{ cm}$$

$$1 \text{ cm} = 0.5 \text{ unit}$$

Now, measuring the scale

Distance from $(S = 0)$ pole to upper dominant pole; $AO = 3.85 \text{ cm} = 1.925 \text{ unit}$

Distance from $(S = -4)$ pole to upper dominant pole; $AB = 5.9 \text{ cm} = 2.95 \text{ unit}$

Distance from $(S = -5)$ pole to upper dominant pole; $AC = 7.8 \text{ cm} = 3.9 \text{ unit}$

Now,

$$K = \frac{\text{Multiplication of distance of every pole to dominant pole}}{\text{Multiplication of distance of every zero to dominant pole}}$$

$$= AO \times AB \times AC = 1.925 \times 2.95 \times 3.9$$

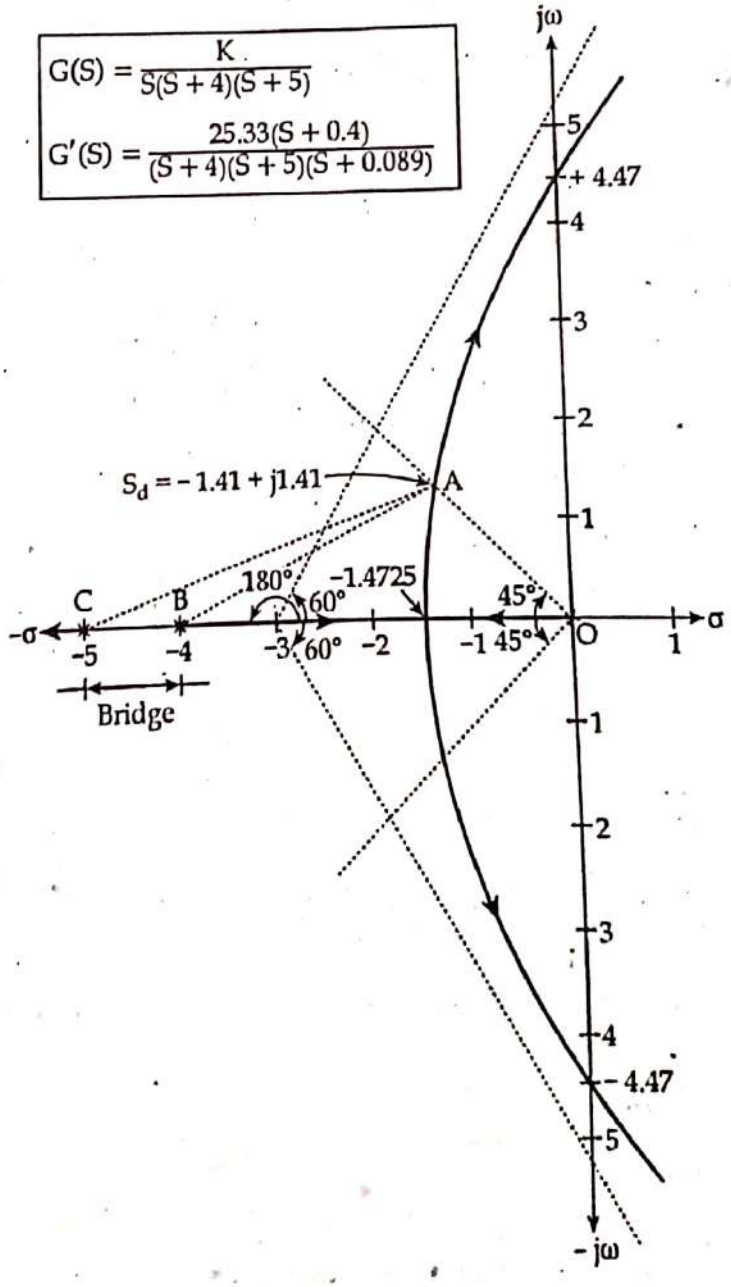
$$= 22.147 \approx 22.15$$

Step 4 Transfer function of uncompensated system

$$G(S) = \frac{22.15}{S(S+4)(S+5)} \text{ for } \zeta = 0.707$$

$$G(S) = \frac{K}{S(S+4)(S+5)}$$

$$G'(S) = \frac{25.33(S+0.4)}{(S+4)(S+5)(S+0.089)}$$



Step 5

$$K_{v\,(new)} = 5 \text{ sec}^{-1}$$

$$K_{v\,(old)} = \lim_{S \longrightarrow 0} SG(S)$$

$$= \lim_{S \longrightarrow 0} S\left[\frac{22.15}{S(S+4)(S+5)}\right] = \frac{2.15}{20}$$

$$= 1.1075$$

Hence,

$$\alpha = \frac{K_{v\,(old)}}{K_{v\,(new)}} = \frac{1.1075}{5} = 0.2215$$

Step 6 Transfer function of compensator

$$G_c(S) = \frac{K_C\left(S + \frac{1}{\alpha T}\right)}{\left(S + \frac{1}{T}\right)}$$

Distance between origin to nearest first pole $(S = -4)$ is 4 unit and give the location of zero.

So, $$10\% \text{ of this distance} = \frac{1}{\alpha T}$$

Or, $$\frac{1}{\alpha T} = 4 \times 10\%$$

Or, $$\frac{1}{\alpha T} = 0.4$$

And, from this we can find out the value of $\left(\frac{1}{T}\right)$

Scanned with CamScanner

and, $\frac{1}{\tau} = 0.2215 \times 0.4 = 0.0886$

Step 7 Overall transfer function of compensator

$$G_c(S) = \frac{K_C(S+0.4)}{(S+0.0886)}$$

Given that;

Undamped frequency $(\omega_n) = 2$ rad/sec.

Step 8

Dominant pole through which new compensated system passes through;

$$S_d = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -(0.707) \times 2 \pm j2\sqrt{1-(0.707)^2} = -1.414 \pm j1.414$$

Step 9 Overall transfer function

$$G'(S) = G_c(S) \times G(S) = \frac{22.15K_C(S+0.4)}{S(S+4)(S+5)(S+0.0886)}$$

Step 10

Now, using magnitude criteria; we have,

$$|G'(S)|_{S_d} = 1$$

or, $$\left|\frac{22.15K_C(S+0.4)}{S(S+4)(S+5)(S+0.0886)}\right|_{S_d=-1.414+j1.414} = 1$$

or, $$\frac{22.15 \times K_C \times 1.74}{2 \times 2.95 \times 3.855 \times 1.938} = 1$$

or, $K_C = 1.1437$

Step 11

The overall transfer function for a compensated system is given by;

$$G'(S) = \frac{22.15 \times 1.1437 \times (S+0.4)}{S(S+4)(S+5)(S+0.0886)} = \frac{25.333(S+0.4)}{S(S+4)(S+5)(S+0.0886)}$$

Step 12

To check whether the velocity error constant match or not

$$K_v = \lim_{S \to 0} SG'(S) = \lim_{S \to 0} S\left[\frac{25.333(S+0.4)}{S(S+4)(S+5)(S+0.0886)}\right] = \frac{25.333 \times 0.4}{4 \times 5 \times 0.0886} = 5.71 \approx 5 \text{ sec}^{-1}$$

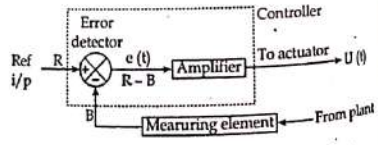
$$\therefore \quad \boxed{K_v = 5.71 \text{ sec}^{-1} \approx 5 \text{ sec}^{-1}}$$

7.6 PID CONTROLLERS

7.6.1 Controllers (Compensator)

The classification is based on control action. There are six controllers which are as follows:

i) P-controller
ii) I-controller
iii) D-controller
iv) PI-controller
v) PD-controller
vi) PID-controller



7.6.1.1 Proportional controller (P – controller)

The output U(t) is proportional to input error signal.

i.e., $U(t) \propto e(t)$

Taking Laplace transform; we have;

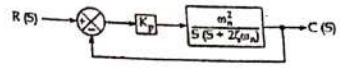
$$U(S) = K_pE(S)$$

where, K_p is the proportional controller gain.

Considering the second order system with proportional controller as shown; we have,

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + 2\zeta\omega_nS + \omega_n^2} \text{ before controller}$$

and, $$\frac{C(S)}{R(S)} = \frac{K_p\omega_n^2}{S^2 + 2\zeta\omega_nS + K_p\omega_n^2}$$



Comparing the characteristics equation with $S^2 + 2\zeta'\omega_n'S + (\omega_n')^2 = 0$; we get,

$\omega_n' = K_p\omega_n^2$; which implies $\omega_n' = \omega_n\sqrt{K_p}$

and, $2\zeta'\omega_n' = 2\zeta\omega_n$

or, $$\zeta' = \left(\frac{\omega_n}{\omega_n'}\right)\zeta = \left(\frac{\omega_n}{\omega_n\sqrt{K_p}}\right)\zeta = \left(\frac{\zeta}{\sqrt{K_p}}\right)$$

Hence, by the introduction of proportional controller; we can say that,

i) Response becomes faster
with the decrease in ζ', rise time and hence peak time decreases. *i.e.*, $\zeta \downarrow, t_r \downarrow, t_p \downarrow$

Since, $t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

ii) Maximum overshoots increases;

$$M_p = e^{\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\%$$

with the decrease in ζ, M_p increases.

iii) Steady state error (e_{ss}) for unit ramp input decrease and hence accuracy increases.

For unit ramp;

$r(t) = t$

$$R(S) = \frac{1}{S^2}$$

Now,

$$K_v = \lim_{S \to 0} SG(S)H(S)$$

$$= \lim_{S \to 0} S\left[\frac{K_p\omega_n^2}{S(S+2\zeta\omega_n)}\right] = \lim_{S \to 0}\left(\frac{K_p\omega_n^2}{S+2\zeta\omega_n}\right) = \frac{K_p\omega_n^2}{2\zeta\omega_n} = \frac{K_p\omega_n}{2\zeta}$$

Hence,

$$e_{ss} = \frac{1}{K_v} = \frac{2\zeta}{K_p\omega_n}$$

with decrease in ζ, the steady error (e_{ss}) also decreases.

7.6.1.2 Derivative Controller (D–Controller)

The value of controller output U(t) is proportional to the rate of change of actuating error signal input and not to actuating error itself. So, it is not used alone. It is combined either with P–controller or I–controller.

i.e., $$U(t) \propto \frac{de(t)}{dt}$$

Taking Laplace transform; we have,

Scanned with CamScanner

$$U(S) = K_d S E(S)$$

where, K_d derivative controller gain.

7.5.1.3 Integral Controller (I-Controller)

The output of controller $U(t)$ is proportional to the integral of the input error signal.

i.e. $U(t) \propto \int e(t)\,dt$

Taking Laplace transform; we have,

$$U(S) = \frac{K_i}{S} E(S)$$

or, $\frac{U(S)}{E(S)} = \frac{K_i}{S}$

where, K_i is the integral controller gain.

7.5.1.4 PD controller (Lead compensator)

PD controller acts as a lead compensator.

Now, the output $U(t)$ is:

$$U(t) = K_p e(t) + K_p T_d \frac{de(t)}{dt}$$

where, T_d is the amount of lead expressed in unit of time.

Taking Laplace transform; we have,

$$U(S) = K_p E(S) + K_p T_d S E(S)$$

or, $U(S) = K_p[1 + ST_d]E(S)$

or, $\frac{U(S)}{E(S)} = K_p(1 + ST_d)$

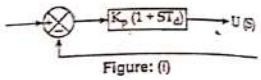


Figure: (i)

Consider a second order system with (P + D) controller [figure (ii)];

$$\frac{C(S)}{R(S)} = \frac{(1+ST_d)\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2 ST_d + \omega_n^2} = \frac{(1+ST_d)\omega_n^2}{S^2 + (2\zeta\omega_n + \omega_n^2 T_d)S + \omega_n^2}$$

Comparing characteristics equation with standard equation $S^2 + 2\zeta'\omega_n' S + (\omega_n')^2$; we have,

$(\omega_n')^2 = \omega_n^2$

$\omega_n' = \omega_n$

and, $2\zeta'\omega_n' = (2\zeta\omega_n + T_d\omega_n^2) = 2\omega_n\left(\zeta + \frac{\omega_n T_d}{2}\right)$

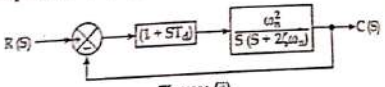


Figure: (ii)

Since, $\omega_n = \omega_n'$

so, $\zeta' = \left(\zeta + \frac{T_d}{2}\omega_n\right)$

With the introduction of d-controller, ω_n remains constant and ζ gets increased.

Hence, for ramp input;

$$K_v = \lim_{S \to 0} SG(S)H(S) = \lim_{S \to 0}\left[S(1+ST_d)\frac{\omega_n^2}{S(S+2\zeta\omega_n)}\right] = \lim_{S \to 0}\frac{(1+ST_d)\omega_n^2}{S+2\zeta\omega_n}$$

$$= \frac{\omega_n^2}{2\zeta\omega_n} = \frac{\omega_n}{2\zeta}$$

so, Steady state error $(e_{ss}) = \frac{1}{K_v} = \frac{2\zeta}{\omega_n}$

Thus, steady state error increases with ζ increasing.

Characteristics of PD-controller

i) Damping coefficient increase ($\zeta \uparrow$)

ii) Rise time (t_r) and peak time reduces ($t_r, t_p \downarrow$)

iii) Maximum overshoot (M_p) decreases ($M_p \downarrow$) i.e., system becomes non-oscillatory that means overshoot decreases.

iv) Improves the bandwidth

v) Response is stable

Derivate feedback control

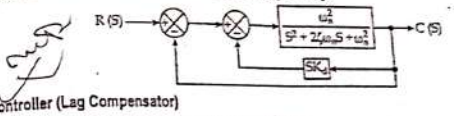
The derivative feedback control is also called **tachometer feedback control**. In this case, the actuating error signal is obtained as the difference between the proportional error signal and derivative of output signal. The actuating error signal for derivative feedback control action is given by;

$$e_a(t) = e(t) - K_d\frac{dC(t)}{dt}$$

Taking Laplace transform; we have,

$$E_a(S) = E(S) - SK_d(S)$$

The block diagram for derivative feedback control is given by;



7.5.1.5 PI-Controller (Lag Compensator)

It acts as a lag compensator. The output U(t) is given by;

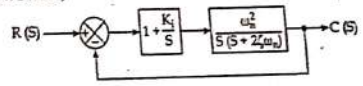
$$U(t) = K_p e(t) + K_i K_p \int e(t)\,dt$$

Taking Laplace transform; we have,

$$U(S) = K_p\left(1 + \frac{K_i}{S}\right)E(S)$$

or, $\frac{U(S)}{E(S)} = K_p\left(1 + \frac{K_i}{S}\right)$

For second order system; we have,



Here,

$$\frac{C(S)}{R(S)} = \frac{(S+K_i)\omega_n^2}{S^3 + 2\zeta\omega_n S^2 + \omega_n^2 S + K_i\omega_n^2}$$

Now, for ramp input;

$$K_v = \lim_{S \to 0} SG(S)H(S) = \lim_{S \to 0}\left[S \times \frac{(S+K_i)\omega_n^2}{S^2(S+2\zeta\omega_n)}\right] = \infty$$

and, Steady state error $(e_{ss}) = \frac{1}{K_v} = \frac{1}{\infty} = 0$

and, for unit parabolic input;

$$K_a = \lim_{S \to 0} S^2 G(S)H(S) = \lim_{S \to 0}\left[S^2 \times \frac{(S+K_i)\omega_n^2}{S^2(S+2\zeta\omega_n)}\right] = \lim_{S \to 0}\left[\frac{(S+K_i)\omega_n^2}{(S+2\zeta\omega_n)}\right] = \frac{K_i\omega_n^2}{2\zeta\omega_n} = \frac{K_i\omega_n}{2\zeta}$$

and, $e_{ss} = \frac{1}{K_a} = \frac{2\zeta}{K_i\omega_n}$

Therefore, with the introduction of I-controller, e_{ss} for unit ramp input becomes zero.

Scanned with CamScanner

Characteristics of PI controller

i) Steady state error improves
ii) Rise time, peak time increases
iii) Bandwidth decreases
iv) Maximum overshoot increases hence response is oscillatory.

7.6.1.6 PID Controller

PID controller is the combination of all proportional, derivative and integral controller.

Here,

$U(t) = K_p e(t)$; for proportional

$U(t) = K_p T_d \left[\frac{de(t)}{dt}\right]$; for derivative

$U(t) = K_p K_i \int_0^t e(t)\,dt = \frac{K_p}{T_i}\int_0^t e(t)\,dt$; for integral controller

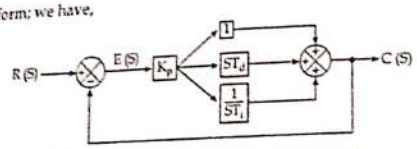
where, $T_i = \frac{1}{K_i}$ = Integral time

and $\frac{1}{T_i}$ = Reset rate

Now,

$$U(t) = K_p e(t) + K_p T_d \left[\frac{de(t)}{dt}\right] + \frac{K_p}{T_i}\int_0^t e(t)\,dt$$

Taking Laplace transform; we have,

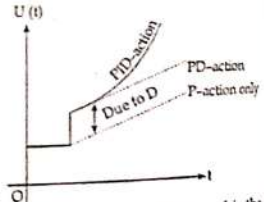


$$\frac{U(S)}{E(S)} = K_p\left(1 + ST_d + \frac{1}{ST_i}\right)$$

Let, $e(t) = At$; then,

$$U(t) = K_p At + K_p T_d\left[\frac{d}{dt}(At)\right] + \frac{K_p}{T_i}\int_0^t At\,dt$$

or, $$U(t) = K_p A\left(t + \frac{t^2}{2T_i} + T_d\right)$$



It uses the property of both PD and PI controller *i.e.*, it improves steady state as well as transient response of the system to meet our desired specifications. To do this the value of K_p, T_i and T_d should be tuned correctly.

From above equation, the proportional part of the control action repeats the change of error. (Lower straight line)

Derivative part of the control action adds an increment of output so that PD action is shifted ahead in time (middle straight line).

The integral part adds further increment part of output proportional to the area under deviation line.

The characteristics of PID control action are:

i) No oscillation ($\zeta < 1$)
ii) Improves the transient response
iii) Improves the steady state response

7.7 SOLVED PROBLEMS

1. Certain measurement was conducted on a servomechanism which shows the system response as; $C(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$ when subjected to a unit step input.
   i) Find the expression for closed loop transfer function.
   ii) Obtained the undamped natural frequency and damping ratio of the system.

Solution:

Here, $C(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$ is output. Let, $r(t) = 1$ be the input.

Taking Laplace transform of input r(t);

$$R(S) = \frac{1}{S}$$

Taking Laplace transform of output C(t);

$$C(S) = \frac{1}{S} + \frac{0.2}{S+60} - \frac{1.2}{S+10} = \frac{(S+10)(S+60) + 0.2S(S+10) - 1.2S(S+60)}{S(S+10)(S+60)}$$

$$= \frac{\left(S^2 + 70S + 600\right) + \left(0.2S^2 + 2S\right) - \left(1.2S^2 + 72S\right)}{S(S+10)(S+60)} = \frac{600}{S(S+10)(S+60)}$$

i) Closed loop transfer function $= \frac{C(S)}{R(S)} = \frac{\frac{600}{S(S+10)(S+60)}}{\frac{1}{S}} = \frac{600}{(S+10)(S+60)} = \frac{600}{S^2+70S+600}$

ii) We know that;

Transfer function of second degree equation $= \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$

Comparing characteristics equation $S^2 + 2\zeta\omega_n S + \omega_n^2$ with $S^2 + 70S + 600$; we get,

$\omega_n^2 = 600$

$\therefore$ Undamped natural frequency $(\omega_n) = \sqrt{600} = 24.5$ rad/sec.

Again,

$2\zeta\omega_n = 70$

or, $\zeta\omega_n = 35$

$\therefore$ Damping coefficient $(\zeta) = \frac{35}{24.5} = 1.429 \approx 1.43$

2. **Obtain the unit step response of a unity feedback system whose open loop transfer function is $G(S) = \frac{4}{S(S+5)}$.**

Solution:

$H(S) = 1$; since, unity feedback system

$$G(S) = \frac{4}{S(S+5)}$$

Now, we have to find the transfer function;

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)H(S)} = \frac{\frac{4}{S(S+5)}}{1 + \frac{4}{S(S+5)}} = \frac{4}{S^2+5S+4}$$

For unit step response $R(S) = \frac{1}{S}$

Therefore, the output is;

Scanned with CamScanner

$$C(S) = R(S) \times \frac{4}{S^2+5S+4} = \frac{4}{S(S^2+5S+4)} = \frac{4}{S(S+1)(S+4)}$$

Now, response becomes; $C(S) = \frac{4}{S(S+1)(S+4)}$

Partial fraction;

$$\frac{4}{S(S+1)(S+4)} = \frac{A}{S} + \frac{B}{S+1} + \frac{C}{S+4}$$

Hence,

$$A = \left.\frac{4}{(S+1)(S+4)}\right|_{S=0} = \frac{4}{4} = 1$$

$$B = \left.\frac{4}{S(S+4)}\right|_{S=-1} = \frac{4}{3\times(-1)} = -\left(\frac{4}{3}\right)$$

$$C = \left.\frac{4}{S(S+1)}\right|_{S=-4} = \frac{4}{(-4)\times(-3)} = \left(\frac{1}{3}\right)$$

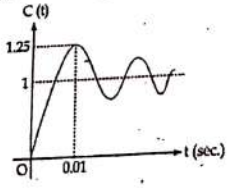
so, $\frac{4}{S(S+1)(S+4)} = \frac{1}{S} - \frac{4}{3(S+1)} + \frac{1}{3(S+4)}$

or, $C(S) = \frac{1}{S} - \frac{4}{3(S+1)} + \frac{1}{3(S+4)}$

Taking inverse Laplace transform; we have,

$C(t) = 1 - \frac{4}{3}e^{-t} + \frac{1}{3}e^{-4t}$; which is the required response.

3. **The unit step response of a linear control system is shown in figure below. Find the transfer function of a second order system to model the system.**



Solution:

Here,

Peak time $(t_p) = 0.01$

Maximum overshoot $= 1.25 - 1 = 0.25$

Now,

$$\text{Peak time } (t_p) = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$$

For first peak overshoot n = 1; so,

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\text{Maximum overshoot } (M_p) = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$$

or, $M_p = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$

or, $0.25 = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$

Taking $\log_e$ on both sides; we have,

$$ln(0.25) = -\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

or, $-1.38 = -\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$

Squaring both sides; we have,

$$1.9218 = \frac{\pi^2\zeta^2}{1-\zeta^2}$$

or, $1.9218 - 1.9218\zeta^2 = \pi^2\zeta^2$

or, $1.9218 = (\pi^2 + 1.9218)\zeta^2$

or, $\zeta^2 = \frac{1.9218}{\pi^2 + 1.9218}$

or, $\zeta^2 = 0.163$

$\therefore \zeta = 0.4037$

Now, substituting in the expression of peak time; we have,

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

or, $0.01 = \frac{\pi}{\omega_n\sqrt{1-0.163}}$

$\therefore \omega_n = 343.38$ rad/sec.

Now,

$$\text{Transfer function for second order system} = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

$$= \frac{117914.1646}{S^2 + 2\times0.4\times343.38\,S + 117914.1646}$$

$\therefore$ $\boxed{\frac{C(S)}{R(S)} = \frac{117914.165}{S^2 + 277.25\,S + 117914.165}}$

4. **The system below in figure (a) when subjected to a unit step input, the output response is as shown in the figure (b). Determine the value of K and T.** [2073 Chaitra]

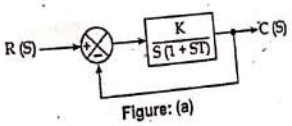


Figure: (a)

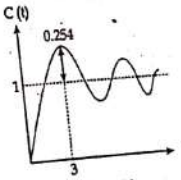


Figure: (b)

Scanned with CamScanner

Solution:

Given that;

Maximum overshoot $(M_p) = 0.254$

Peak time $(t_p) = 3$ sec.

We know that;

$$M_p = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$$

or, $0.254 = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$

Taking natural log on both sides; we have,

$$\ln(0.254) = -\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

or, $1.37 = \frac{\pi\zeta}{\sqrt{1-\zeta^2}}$

Squaring both sides; we have,

$$(1.37)^2 = \frac{\pi^2\zeta^2}{1-\zeta^2}$$

or, $1.878 - 1.878\zeta^2 = \pi^2\zeta^2$

or, $1.878 = (1.878 + \pi^2)\zeta^2$

or, $\zeta^2 = \frac{1.878}{1.878 + \pi^2}$

or, $\zeta^2 = 0.1598$

∴ Damping coefficient $(\zeta) = 0.399 = 0.4$

Now, substituting value of (ζ) in peak time expression; we have,

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

or, $3 = \frac{\pi}{\omega_n\sqrt{1-(0.4)^2}}$

∴ Undamped natural frequency $(\omega_n) = \frac{\pi}{3 \times 0.917} = 1.1425$ rad/sec.

so, Transfer function $\left[\frac{C(S)}{R(S)}\right] = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} = \frac{1.30}{S^2 + 2 \times 0.4 \times 1.14\,S + 1.30} = \frac{1.30}{S^2 + 0.911\,S + 1.30}$ (1)

From the figure (a); we have,

Transfer function $\left[\frac{C(S)}{R(S)}\right] = \frac{G(S)}{1 + G(S)H(S)} = \frac{\frac{K}{S(1+ST)}}{1 + \frac{K}{S(1+ST)}} = \frac{K}{S(1+ST) + K} = \frac{K}{TS^2 + S + K}$

or, $\frac{C(S)}{R(S)} = \frac{\frac{K}{T}}{S^2 + \left(\frac{1}{T}\right)S + \left(\frac{K}{T}\right)}$ (2)

Comparing (1) and (2) characteristics equation; we get,

$\frac{1}{T} = 0.911$

∴ $\boxed{T = 1.096 \text{ sec.}}$

$\frac{K}{T} = 1.3$

$\boxed{K = 1.3 \times 1.096 = 1.426}$

5. **Following figure shows a mechanical system and the response when 10 N of force is applied to the system. Determine the value of M, D and K. The displacement x is measured from the equilibrium position.** [2071 Chaitra, 2072 Kartik]

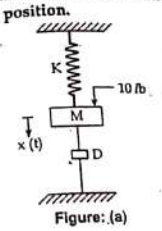


Figure: (a)

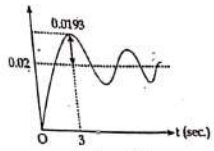


Figure: (b)

Solution:

Here,

Maximum overshoot $(M_p) = 0.0193$

Peak time $(t_p) = 3$ sec.

Now,

$$M_p = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$$

or, $0.0193 = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$

Taking natural log on both sides; we have,

$$\ln(0.0193) = -\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

or, $-3.947 = -\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$

Squaring both sides; we have,

$$15.584 = \frac{\pi^2\zeta^2}{1-\zeta^2}$$

or, $15.584 = (15.584 + \pi^2)\zeta^2$

or, $\zeta^2 = \frac{15.584}{15.584 + \pi^2}$

or, $\zeta^2 = 0.6122$

∴ Damping coefficient $(\zeta) = 0.7825$

Now,

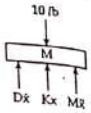
$$\text{Peak time } (t_p) = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

or, $3 = \frac{\pi}{\omega_n\sqrt{1-0.6122}}$

∴ Undamped natural frequency $(\omega_n) = 1.6817$ rad/sec.

Scanned with CamScanner

Now, for a mechanical system; F.B.D. of M becomes as shown aside.



From De-Alembert principle; we have,

$$10 = M\ddot{x} + D\dot{x} + Kx$$

Taking Laplace transform; we have,

$$\frac{10}{S} = (S^2M + SD + K)x(S)$$

and, Input $[R(S)] = \frac{10}{S}$

so, $$\frac{x(S)}{R(S)} = \frac{1}{S^2M + SD + K} = \frac{\frac{1}{M}}{\left[S^2 + \left(\frac{D}{M}\right)S + \left(\frac{K}{M}\right)\right]} = \frac{\frac{1}{M}}{S^2 + \left(\frac{D}{M}\right)S + \left(\frac{K}{M}\right)}$$

Comparing denominator with characteristics equation $S^2 + 2\zeta\omega_n + \omega_n^2 = 0$; we have,

$$\omega_n^2 = \frac{K}{M}$$

or, $$\omega_n = \sqrt{\frac{K}{M}}$$

Squaring both sides; we have,

$$(1.6817)^2 = \left(\sqrt{\frac{K}{M}}\right)^2 = \frac{K}{M} \quad (1)$$

and, $$2\zeta\omega_n = \frac{D}{M}$$

or, $$2 \times 0.7825 \times 1.6817 = \frac{D}{M} \quad (2)$$

or, $$\frac{D}{M} = 2.6319$$

The steady state value is;

$$X(\infty) = 0.02$$

We know that;

$$X(\infty) = \lim_{S \to 0} SX(S)$$

or, $$0.02 = \lim_{S \to 0} S\left(\frac{\frac{10}{S}}{MS^2 + DS + K}\right) = \lim_{S \to 0}\left(\frac{10}{MS^2 + DS + K}\right) = \frac{10}{K}$$

$$K = \frac{10}{0.02} = 500$$

Now, equation (1) becomes;

$$\frac{K}{M} = (1.6817)^2$$

$$M = \frac{500}{2.83} = 176.8 \text{ lb}$$

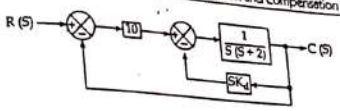
And, equation (2) becomes;

$$\frac{D}{M} = 2.6319$$

$$D = 176.8 \times 2.6319 = 465.31 \text{ lbs/feet}$$

i) If $K_d = 0$, find ζ, ω_n and e_{ss} for unit ramp input.

ii) If damping factor is to be increased to 0.6, find K_d, and e_{ss} for unit ramp input.



Solution:

i) If $K_d = 0$; then,

$$G(S) = 10 \times \frac{1}{S(S+2)} = \frac{10}{S(S+2)}$$

and, $H(S) = 1$

The characteristics equation is;

$$1 + \frac{10}{S(S+2)} \times 1 = 0$$

or, $$S(S+2) + 10 = 0$$

or, $$S^2 + 2S + 10 = 0 \quad (1)$$

Comparing with second degree characteristics equation; we have,

$$S^2 + 2\zeta\omega_n S + \omega_n^2 = 0 \quad (2)$$

Comparing; we have,

$$\omega_n^2 = 10$$

$\therefore$ Undamped natural frequency $(\omega_n) = \sqrt{10} = 3.16$ rad/sec.

Similarly,

$$2\zeta\omega_n = 2$$

or, $$\zeta\omega_n = 1$$

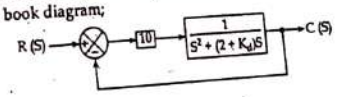
$\therefore$ Damping coefficient $(\zeta) = \frac{1}{\omega_n} = \frac{1}{\sqrt{10}} = 0.316$

ii) For unit ramp input;

$$r(t) = t$$

So, we have to calculate velocity error constant (K_v).

Now, reducing the book diagram;



Now,

$$G(S) = \frac{10}{S^2 + (2 + K_d)S}$$

and, $H(S) = 1$

so, $$\frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)H(S)} = \frac{\frac{10}{S^2 + (2+K_d)S}}{1 + \frac{10}{S^2 + (2+K_d)S}} = \frac{10}{S^2 + (2+K_d)S + 10}$$

Comparing with;

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

so, $$\omega_n^2 = 10$$

or, $$\omega_n = \sqrt{10} \text{ rad/sec.}$$

and, $$2\zeta\omega_n = 2 + K_d$$

or, $2 \times 0.6 \times \sqrt{10} = 2 + K_d$

$\therefore \quad K_d = 1.794$

Now, for steady state error;

$$e_{ss} = \frac{1}{K_v}$$

where, $K_v = \lim_{S \to 0} SG(S)H(S)$.

For case I

When $K_d = 0$; then,

$$G(S)H(S) = \frac{10}{S(S+2)}$$

$$\therefore \quad K_v = \lim_{S \to 0} S\left[\frac{10}{S(S+2)}\right] = 10 \text{ sec}^{-1}$$

When $K_d = 0$;

$$e_{ss} = \frac{1}{K_v} = \frac{1}{10} = 0.1$$

$\therefore$ Steady state error $(e_{ss}) = 0.1$

For case II

$$G(S)H(S) = \frac{10}{S^2 + (2 + K_d)S} = \frac{10}{S^2 + 3.794\,S} = \frac{10}{S(S+3.794)}$$

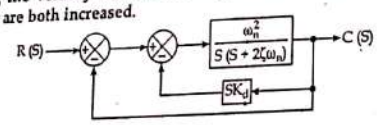
Now,

$$K_v = \lim_{S \to 0} SG(S)H(S) = \lim_{S \to 0} S\left[\frac{10}{S(S+3.794)}\right] = \frac{10}{3.794} = 2.636 \text{ sec}^{-1}$$

Now,

Steady state error $(e_{ss}) = \frac{1}{K_v} = \frac{1}{2.636} = 0.38$

7. **Show that using the velocity feedback techniques shown figure below damping ratio and steady state error are both increased.** [2071 Chaitra]



Solution:

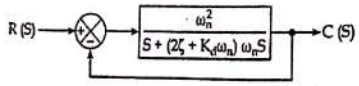
Above diagram is derivative feedback control or sometimes called tachometer feedback control.

Simplifying the block diagram; we have,

$$G(S) = \frac{\frac{\omega_n^2}{S(S+2\zeta\omega_n)}}{1 + \frac{\omega_n^2}{S(S+2\zeta\omega_n)} \times SK_d} = \frac{\omega_n^2}{S(S+2\zeta\omega_n) + SK_d\omega_n^2} = \frac{\omega_n^2}{S^2 + (2\zeta\omega_n + K_d\omega_n^2)S}$$

$$= \frac{\omega_n^2}{S^2 + (2\zeta + K_d\omega_n)\omega_n S}$$

e block diagram becomes,



Now,

$$\text{Transfer function} = \frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)H(S)} = \frac{\omega_n^2}{S^2 + (2\zeta\omega_n + K_d\omega_n^2)S + \omega_n^2} = \frac{\omega_n^2}{S^2 + (2\zeta + K_d\omega_n)\omega_n S + \omega_n^2}$$

and, hence characteristics equation is;

$$S^2 + 2\left(\zeta + \frac{K_d\omega_n}{2}\right)\omega_n S + \omega_n^2 = 0$$

Comparing with $S^2 + 2\zeta'\omega_n S + \omega_n^2 = 0$; we have,

$$\zeta' = \zeta + \frac{\omega_n K_d}{2}$$

Hence, derivative feedback control increases the damping coefficient.

Steady state for unit ramp input is given by;

$$e_{ss} = \frac{1}{K_v}$$

where, K_v is the velocity error constant for unit ramp input.

$$K_v = \lim_{S \to 0} SG(S)H(S)$$

$$= \lim_{S \to 0} S\left[\frac{\omega_n^2}{S^2 + (2\zeta + K_d\omega_n)\omega_n S}\right]$$

$$= \lim_{S \to 0}\left[\frac{\omega_n^2}{S + (2\zeta + K_d\omega_n)\omega_n}\right] = \frac{\omega_n}{(2\zeta + K_d\omega_n)}; \text{ for derivative feedback controller}$$

Hence,

$$e_{ss} = \frac{1}{K_v} = \frac{2\zeta + K_d\omega_n}{\omega_n} = \frac{2\zeta}{\omega_n} + K_d$$

In absence of derivative feedback controller $K_d = 0$; so,

$$G(S) = \frac{\omega_n^2}{S(S + 2\zeta\omega_n)}$$

Hence,

$$K_v = \lim_{S \to 0} SG(S)H(S) = \lim_{S \to 0} S\left[\frac{\omega_n^2}{S(S+2\zeta\omega_n)}\right] = \frac{\omega_n^2}{2\zeta\omega_n} = \frac{\omega_n}{2\zeta}$$

Steady state error $(e_{ss}) = \frac{1}{K_v} = \frac{2\zeta}{\omega_n}$

so, Error without controller $(e_{ss}) = \frac{2\zeta}{\omega_n}$

Error with controller $(e_{ss}) = \frac{2\zeta}{\omega_n} + K_d$

Hence, derivative feedback control increases the steady state error.

8. **For a Mechanical system with close loop transfer function $\frac{\theta(S)}{T(S)} = \frac{1}{aS^2 + bS + C}$; where, $\theta(S)$ the output of step input $T(t) = 10$ Nm. Determine values of a, b, and c; if maximum overshoot is 6%, peak time $(t_p) = 1$ sec. and $e_{ss} = 0.5$** [2070 Ashadh]

Solution:

Given that;

Step input $T(t) = 10$ Nm

Taking Laplace transform; we have,

Scanned with CamScanner

$$T(S) = \frac{10}{S}$$

and, hence,

$$\theta(S) = \frac{10}{S(aS^2 + bS + C)}$$

Steady state value is given by;

$$0.5 = \lim_{S \to 0} S\theta(S) = \lim_{S \to 0} S\left[\frac{10}{S(aS^2 + bS + C)}\right]$$

or, $0.5 = \frac{10}{C}$

$\therefore$ $\boxed{C = 20}$

and, Maximum overshoot $(M_p) = 6\% = 0.06$

so, $M_p = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$

or, $0.06 = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$

Taking natural log on both sides; we have,

$$\ln(0.06) = -\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

or, $2.81 = \frac{\pi\zeta}{\sqrt{1-\zeta^2}}$

Squaring both sides; we have,

$$7.91(1-\zeta^2) = \pi^2\zeta^2$$

or, $7.91 = \zeta^2(\pi^2 + 7.91)$

or, $\zeta^2 = 0.44$

$\therefore$ $\zeta = 0.667$

and, Peak time $(t_p) = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

or, $1 = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

Squaring both sides; we have,

$$\omega_n^2(1-\zeta^2) = \pi^2$$

or, $\omega_n^2 = \frac{\pi^2}{1-\zeta^2} = 17.78$

$\therefore$ $\omega_n = 4.217$ rad/sec.

The characteristics equation of the system is;

$$aS^2 + bS + C = 0$$

or, $a\left[S^2 + \left(\frac{b}{a}\right)S + \left(\frac{c}{a}\right)\right] = 0$

or, $S^2 + \left(\frac{b}{a}\right)S + \left(\frac{c}{a}\right) = 0$

Comparing with $S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$; we have,

$$\omega_n^2 = \frac{c}{a}$$

$\therefore$ $a = \frac{c}{\omega_n^2} = \frac{20}{17.78} = 1.124$

and, $2\zeta\omega_n = \frac{b}{a}$

$\therefore$ $b = 2a\zeta\omega_n = 2 \times 1.124 \times 0.667 \times 4.217 = 6.3232$

9. **The open loop transfer function for a unity feedback system is given by $G(S) = \frac{K}{S(1+ST)}$ where, K is gain constant and T is time constants. With the gain multiplied by a factor K_1 the maximum overshoot of the system is increased from 25% to 50%. Determine K_1. [2067 Ashadh]**

Solution:

We have,

$$G(S) = \frac{K}{S(1+ST)}$$

and, $H(S) = 1$

The characteristics equation is;

$$1 + G(S)H(S) = 0$$

or, $1 + \frac{K}{S(1+ST)} = 0$

or, $TS^2 + S + K = 0$

or, $T\left(S^2 + \frac{S}{T} + \frac{K}{T}\right) = 0$

or, $S^2 + \left(\frac{1}{T}\right)S + \frac{K}{T} = 0$

Comparing with $S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$; we get,

$$\omega_n^2 = \frac{K}{T}$$

or, $\omega_n = \sqrt{\frac{K}{T}}$ (1)

and, $2\zeta\omega_n = \frac{1}{T}$

or, $2\zeta \times \sqrt{\frac{K}{T}} = \frac{1}{T}$

or, $\zeta = \frac{1}{2}\sqrt{\frac{1}{KT}}$

or, $\zeta = \frac{1}{2\sqrt{KT}}$ (2)

At initial condition;

$$M_p = 25\% = 0.25$$

and, $M_p = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$

or, $0.25 = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$

Taking natural log on both sides; we have,

Scanned with CamScanner

$$\ln(0.25) = -\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

or, $$-1.386 = -\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

Squaring both sides; we have,

$$1.9218(1-\zeta^2) = \pi^2\zeta^2$$

or, $$1.9218 = \zeta^2(\pi^2 + 1.9218)$$

or, $$\zeta^2 = \frac{1.9218}{\pi^2 + 1.9218} = 0.163$$

$\therefore$ $$\zeta = 0.4037 = \zeta_{initial}$$

This is the value of ζ before multiplying.

After multiplying by K_1; we have,

$$M_p = 50\% = 0.5$$

or, $$0.5 = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$$

Taking natural log on both sides; we have,

or, $$\ln(0.5) = -\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

or, $$0.48(1-\zeta^2) = \pi^2\zeta^2$$

or, $$0.48 = \zeta^2(\pi^2 + 0.48)$$

or, $$\zeta^2 = \frac{0.48}{\pi^2 + 0.48} = 0.0464$$

$\therefore$ $$\zeta = 0.2154 = \zeta_{final}$$

From equation (2); we have,

$$\zeta_{initial} = \frac{1}{2\sqrt{KT}} \quad (3)$$

and, $$\zeta_{final} = \frac{1}{2\sqrt{KK_1T}} \quad (4)$$

Dividing equation (3) by (4); we get,

$$\frac{\zeta_{initial}}{\zeta_{final}} = \sqrt{\frac{KK_1T}{KT}}$$

or, $$\sqrt{K_1} = \frac{0.4037}{0.2154} = 1.874$$

Squaring both sides; we get,

$$K_1 = 3.5125$$

10. Find unit step response of a system represented by $G(S) = \frac{10}{0.01S^2 + 0.25S + 1}$. [2066 Chaitra]

Solution:

Here,

$$G(S) = \frac{10}{0.01S^2 + 0.25S + 1}$$

We know that;

Scanned with CamScanner

$$\frac{C(S)}{R(S)} = G(S)$$

and, for unit step response;

$$R(S) = \frac{1}{S}$$

Then,

$$C(S) = \frac{10}{S\left(0.01S^2 + 0.25\,S + 1\right)} = \frac{10}{0.01S\left(S^2 + \frac{0.25}{0.01}S + \frac{1}{0.01}\right)} = \frac{1000}{S\left(S^2 + 25\,S + 100\right)} = \frac{1000}{S(S+20)(S+5)}$$

Partial fraction yields;

$$C(S) = \frac{1000}{S(S+20)(S+5)} = \frac{A}{S} + \frac{B}{(S+5)} + \frac{C}{(S+20)}$$

and, $A = \lim_{S \longrightarrow 0}\left[\frac{1000}{(S+5)(S+20)}\right] = 10$

$$B = \lim_{S \longrightarrow -5}\left[\frac{1000}{S(S+20)}\right] = \frac{1000}{15 \times (-5)} = -\left(\frac{40}{3}\right)$$

$$\therefore C = \lim_{S \longrightarrow -20}\left[\frac{1000}{S(S+5)}\right] = \frac{1000}{-20 \times (-15)} = \left(\frac{10}{3}\right)$$

Then,

$$C(S) = \frac{10}{S} - \frac{40}{3(S+5)} + \frac{10}{3(S+20)}$$

Taking inverse Laplace transform; we have,

$C(t) = 10 - \left(\frac{40}{3}\right)e^{-5t} + \left(\frac{10}{3}\right)e^{-20t}$; which is the required response.

11. **Find K and T so that the unity feedback system with open loop transfer function $G(S)H(S) = \frac{K}{S(S+T)}$ will exhibit damped oscillations with peak overshoot 16% and settling time of 2 second.** [2066 Chaitra]

Solution:

Here,

Maximum overshoot $(M_p) = 16\% = 0.16$

or, $0.16 = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$

Taking natural log; we have,

$$\ln(0.16) = -\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

or, $1.832 = \frac{\pi\zeta}{\sqrt{1-\zeta^2}}$

Squaring both sides; we have,

$3.35(1-\zeta^2) = \pi^2\zeta^2$

or, $3.35 = (\pi^2 + 3.35)\zeta^2$

or, $\zeta^2 = \frac{3.35}{\pi^2 + 3.35} = 0.254$

$\therefore \zeta = 0.5$

Scanned with CamScanner

and, Settling time (t_s) = 2 seconds

We have,

$$t_s = \frac{4}{\zeta\omega_n}$$

or, $$2 = \frac{4}{\zeta\omega_n}$$

$$\therefore \quad \omega_n = \frac{4}{2 \times \zeta} = \frac{2}{0.5} = 3.97 \text{ rad/sec.}$$

Now,

$$G(S)H(S) = \frac{K}{S(S+T)}$$

The characteristics equation is given by;

$$1 + G(S)H(S) = 0$$

or, $$1 + \frac{K}{S(S+T)} = 0$$

or, $$S^2 + ST + K = 0$$

Comparing with $S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$; we get,

$$K = \omega_n^2 = (3.97)^2 = 15.75$$

Similarly;

$$2\zeta\omega_n = T$$

$$\therefore \quad T = 2 \times 3.97 \times 0.5 = 3.97 \text{ sec.}$$

12. A control system has feed forward transfer function $G(S) = \frac{12}{(S^2 + 4S + 16)}$ and feedback transfer function $H(S) = KS$. For damping factor of 0.8; determine the value of K in the transfer function and calculate first undershoot of the system. [2066 Magh]

Solution:

Given that;

$$G(S) = \frac{12}{(S^2 + 4S + 16)}$$

and, $H(S) = KS$

Now,

$$\text{Transfer function} = \frac{G(S)}{1 + G(S)H(S)}; \text{ for negative feedback system}$$

or, $$\text{T.F.} = \frac{C(S)}{R(S)} = \frac{\frac{12}{S^2 + 4S + 16}}{1 + \left(\frac{12}{S^2 + 4S + 16}\right)KS} = \frac{12}{S^2 + 4S + 16 + 12KS} = \frac{12}{S^2 + (4 + 12K)S + 16}$$

We know that, the characteristics equation is;

$$S^2 + 4(1 + 3K)S + 16 = 0$$

Comparing with $S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$; we get,

$$\omega_n^2 = 16$$

$$\therefore \quad \omega_n = 16 = 4 \text{ rad/sec.}$$

Form the question; we have,

$$2\zeta\omega_n = (4 + 12K)$$

Scanned with CamScanner

$2 \times 0.8 \times 4 = 4 + 12K$

or, $12K = 2.4$

or, $K = 0.2$

We have,

$$\text{Peak time } (t_p) = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$$

For first peak undershoot; take $n = 2$,

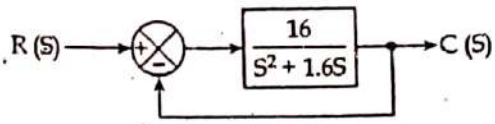
$$t_p = \frac{2\pi}{4\sqrt{1-(0.8)^2}} = 2.618 \text{ sec.}$$

This is the time taken to reach first peak undershoot.

The maximum overshoot can be found by;

$$M_p = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)} \times 100\% = e^{-\left(\frac{\pi \times 0.8}{\sqrt{1-(0.8)^2}}\right)} \times 100\% = 1.51\% = 0.015$$

13. A unity feedback control system is shown in figure. By using derivative control, the damping ration is to be made 0.8. Determine the value of T_d and compare maximum percent overshoot before and after adding derivative controller. The input to system being step input. [2064 Falgun]



Solution:

Here,

$$G(S) = \frac{16}{S^2 + 1.6S}$$

and, $H(S) = 1$

The characteristics equation is given by;

$$1 + G(S)H(S) = 0$$

or, $1 + \frac{16}{S^2 + 1.6S} = 0$

or, $S^2 + 1.6S + 16 = 0$

Comparing with $S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$; we get,

$\omega_n^2 = 16$

$\therefore \omega_n = 4$ rad/sec.

Similarly;

$2\zeta\omega_n = 1.6$

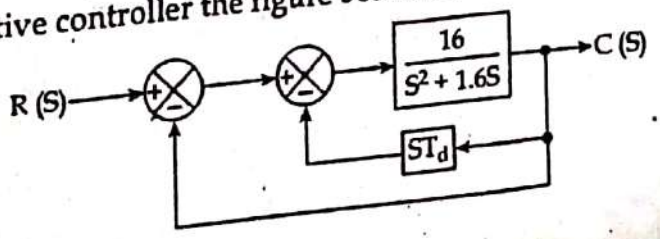
or, $2\zeta \times 4 = 1.6$

$\therefore \zeta = 0.2$

$$\text{Maximum overshoot } (M_p) = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)} = e^{-\left(\frac{\pi \times 0.2}{\sqrt{1-(0.2)^2}}\right)} = 0.5266$$

$= 52.66\%$; before adding controller

After the addition of derivative controller the figure becomes;



Scanned with CamScanner

Reducing inner loop; we get,

$$\text{Forward transfer function} = G(S) = \frac{\frac{16}{S^2+1.6S}}{1+\left(\frac{16}{S^2+1.6S}\right)\times ST_d} = \frac{16}{S^2+(1.6+16T_d)S}$$

Now,

$$\text{T.F.} = \frac{C(S)}{R(S)} = \frac{\frac{16}{S^2+(1.6+1.6T_d)S}}{1+\frac{16}{S^2+(1.6+1.6T_d)S}} = \frac{16}{S^2+(1.6+16T_d)S+16}$$

Now, comparing the characteristics equation $S^2+(1.6+16T_d)S+16=0$ with $S^2+2\zeta\omega_n S+\omega_n^2=0$; we get,

$\omega_n^2 = 16$

$\therefore \quad \omega_n = 4$ rad/sec.

Similarly;

$2\zeta\omega_n = (1.6+16T_d)$

or, $2\times 0.8\times 4 = 1.6+16T_d$

or, $\frac{6.4-1.6}{16} = T_d$

$\therefore \quad T_d = 0.3$ sec.

Maximum overshoot after addition of controller;

$$M_p = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$$

Put, $\zeta = 0.8$; then,

$$\therefore \quad M_p = e^{-\left(\frac{\pi\times 0.8}{\sqrt{1-(0.8)^2}}\right)} = e^{-\left(\frac{2.513}{0.6}\right)} = 0.015 = 1.51\%$$

Hence, for unit step input the maximum percentage overshoot decreases after adding derivation controller.

14. A system with open loop transfer function and unit feedback $G(S) = \frac{K}{S(S+2)}$ represent by following specifications:

i) $\omega_n = 4$ rad/sec.

ii) Damping ration = 0.5

State why only adjusting the K alone cannot meet both requirements? Also suggest the type of compensator needed to achieve the required response (Detail design is not required)

[2062 Baishakh]

Solution:

Here,

$$G(S) = \frac{K}{S(S+2)}$$

and, $H(S) = 1$

The characteristics equation is given by;

$1+G(S)H(S) = 0$

or, $1+\frac{K}{S(S+2)}\times 1 = 0$

or, $S(S+2)+K = 0$

or, $S^2+2S+K = 0$

Scanned with CamScanner

Comparing this with characteristics equation $S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$; we get,

$$\omega_n^2 = K$$

$$\omega_n = \sqrt{K} \quad (1)$$

and, $2\zeta\omega_n = 2$

or, $\zeta\omega_n = 1$

Solving $\omega_n = 4$ rad/sec. gives; (2)

$$K = \omega_n^2 = 16$$

$$K = 16$$

and, $\zeta\omega_n = 1$

$$\zeta \times 4 = 1$$

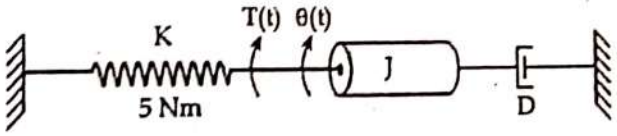
$$\zeta = \frac{1}{4} = 0.25$$

From equation (1) and (2), we can see that ω_n and ζ are co-related and varies simultaneously with the variation of K but not independently. Thus adjustment of gain K alone is not sufficient to achieve both the requirements.

For this we can use lead compensator to meet the angle deficiency at the location of desired poles. The angle to be contributed by such compensator can now be used to determine the location of its pole and zeros with the help of root locus.

Finally, the transfer function of overall compensated system can be determined which eventually leads to fulfillment of desired requirements.

3. **For the system shown find J and D to yield 20% overshoot and settling time of 2 second for a step input of torque T(t). In the figure θ(t) is the output.** **[2066 Bhadra]**



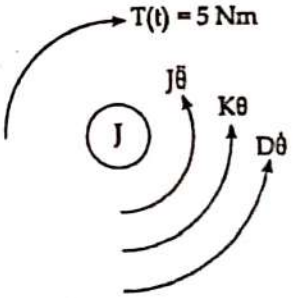
Solution:

Here,

Input torque = T(t)

and, K = 5 Nm

Now, free body diagram of J becomes as shown aside.



From De-Alembert's principle; we have,

$$T(t) = J\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} + K\theta \quad (1)$$

Taking Laplace transform of equation (1); we have,

$$T(S) = [S^2J + SD + K]\theta(S)$$

$$\text{or, } \frac{\theta(S)}{T(S)} = \frac{1}{S^2J + DS + K} = \frac{\frac{1}{J}}{S^2 + \left(\frac{D}{J}\right)S + \left(\frac{K}{J}\right)} \quad (2)$$

Comparing characteristics equation $S^2 + \left(\frac{D}{J}\right)S + \left(\frac{K}{J}\right) = 0$ with $S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$; we have,

$$\omega_n^2 = \frac{K}{J} \quad (3)$$

$$\text{or, } \omega_n = \sqrt{\frac{K}{J}}$$

Scanned with CamScanner

and, $2\zeta\omega_n = \frac{D}{J}$

or, $2\zeta \times \sqrt{\frac{K}{J}} = \frac{D}{J}$ (4)

or, $\zeta = \frac{D}{2\sqrt{KJ}}$

Given that;

$M_p = 20\% = 0.2$

so, $e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)} = 0.2$

or, $\ln(0.2) = -\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$

or, $-1.60 = -\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$

Squaring both sides; we have,

$2.59(1-\zeta^2) = \pi^2\zeta^2$

or, $\frac{2.59}{(2.59+\pi^2)} = \zeta^2$

or, $\zeta^2 = 0.20$

$\therefore \quad \zeta = 0.456$

and, Settling time $(t_s) = 2$ seconds

or, $t_s = \frac{4}{\zeta\omega_n}$

or, $2 = \frac{4}{0.456 \times \omega_n}$

$\therefore \quad \omega_n = 4.386$ rad/sec.

From equation (3), put K = 5 Nm;

$\omega_n^2 = \frac{K}{J}$

$\therefore \quad J = \frac{K}{\omega_n^2} = \frac{5}{19.24} = 0.26$ Nmsec.^{2}rad^{-1}

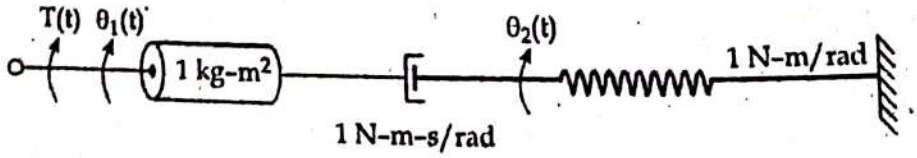
and, equation (4) becomes;

$2\zeta = \frac{D}{\sqrt{KJ}}$

Put value of K, J and ζ; we get,

$D = 2\zeta\sqrt{KJ} = 2 \times 0.456 \times \sqrt{5 \times 0.26} = 1.039$ Nm-sec.

16. A step torque T(t) is applied in a system shown in figure below. Find the percent overshoot, settling time and peak time for output $(\theta_2)t$. [2067 Ashadh]



Solution:

Let, $J_1 = 1$ kg/m^2, B = 1 Nmsec./rad and K = 1 Nm/rad

Scanned with CamScanner

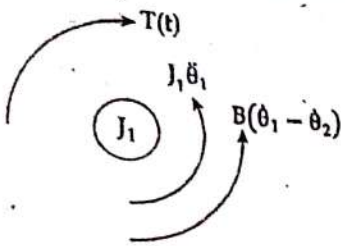


Figure: (i) F.B.D. of J_1

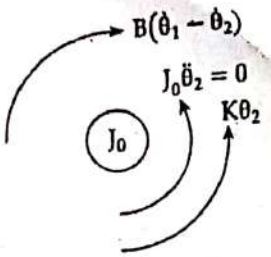


Figure: (ii) F.B.D. of J_0

Now, drawing free body diagram of J_1 and mass-less mass whose moment of inertia J_0 as;

and $J_0 = 0$

Applying De-Alembert's principle on $(J_0 = 0)$;

$$B(\dot{\theta}_1 - \dot{\theta}_2) = K\theta_2$$

or, $$B\dot{\theta}_1 - B\dot{\theta}_2 = K\theta_2 \quad (1)$$

Taking Laplace transform of equation (1); we have,

$$SB\theta_1(S) - SB\theta_2(S) = K\theta_2(S)$$

or, $$\theta_2(S) = \frac{BS}{(BS + K)}\theta_1(S) \quad (2)$$

Applying De-Alembert's principle on J_1; we have,

$$T(t) = J_1\ddot{\theta}_1 + B(\dot{\theta}_1 - \dot{\theta}_2) \quad (3)$$

Applying Laplace transform on equation (3); we have,

$$T(S) = [S^2J_1 + SB]\theta_1(S) - SB\theta_2(S) = [J_1S^2 + SB]\theta_1(S) - SB\theta_2(S) \quad (4)$$

Substituting value of $\theta_1(S)$ in equation (4); we have,

$$T(S) = [J_1S^2 + SB]\left(\frac{BS + K}{BS}\right)\theta_2(S) - \theta_2(S) \times BS = \frac{\left[(J_1S^2 + BS)(BS + K) - B^2S^2\right]\theta_2(S)}{BS}$$

or, $$\frac{\theta_2(S)}{T(S)} = \frac{BS}{(J_1S^2 + BS)(BS + K) - B^2S^2} = \frac{B}{(J_1S + B)(BS + K) - B^2S} \quad (5)$$

The characteristics equation is given by;

$$(J_1S + B)(BS + K) - B^2S = 0$$

or, $$J_1BS^2 + (B^2 + J_1K)S + BK - B^2S = 0$$

or, $$J_1BS^2 + (B^2 + J_1K - B^2)S + BK = 0$$

or, $$J_1BS^2 + J_1KS + BK = 0$$

or, $$J_1B\left[S^2 + \left(\frac{J_1K}{J_1B}\right)S + \frac{BK}{J_1B}\right] = 0$$

or, $$S^2 + \left(\frac{K}{B}\right)S + \frac{K}{J_1} = 0$$

Comparing with $S^2 + 2\zeta\omega_nS + \omega_n^2 = 0$; we get,

$$\omega_n^2 = \frac{K}{J_1} = \frac{1}{1} = 1$$

$$\omega_n = 1 \text{ rad/sec.}$$

and, $$2\zeta\omega_n = \frac{K}{B}$$

Substituting the value of ω_n; we get,

Scanned with CamScanner

$$2\zeta \times 1 = \frac{1}{1} = 1$$

$\therefore \quad \zeta = 0.5$

i) Percent overshoot $(M_p) = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)} = e^{-\left[\frac{\pi \times 0.5}{\sqrt{1-(0.5)^2}}\right]} = e^{-1.81} = 0.1630 = 16.3\%$

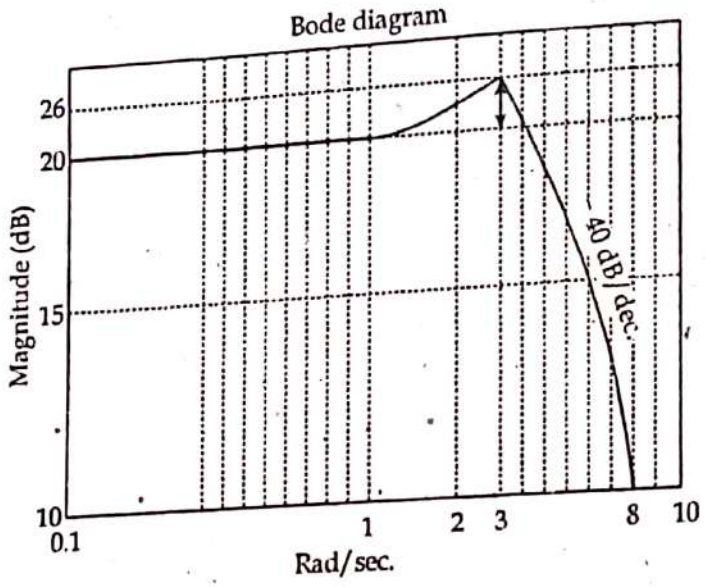
ii) Settling time $(t_s) = \frac{4}{\zeta\omega_n} = \frac{4}{0.5 \times 1} = 8$ sec.

iii) Peak time $(t_p) = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{1 \times \sqrt{1-(0.5)^2}} = 3.62$ sec.

17. For an approximate Bode plot of a closed loop system as shown below, evaluate the following performance indices: [2070 Ashadh]

i) Maximum overshoot

ii) Peak time



Solution:

From Bode diagram; we have,

Maximum overshoot $(M_p) = 26 - 20 = 6$ dB

The transfer function of plot can be calculated by using corner frequencies.

From $\omega = 0.1$ to $\omega = 1$ rad/sec.; the slope is 0 dB/dec; which indicates that transfer function contains Type-zero system.

From $\omega = 3$ rad/sec.; the slope changes to -40 dB/dec this is due to second order equation.

$\therefore \quad \left[1 + \left(\frac{S}{\omega_n}\right)^2 + 2\zeta\left(\frac{S}{\omega_n}\right)\right]$; in pole

Magnitude at $\omega = 3$ rad/sec. $= \omega_n$ is;

$$\text{Magnitude} = \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta}{\omega_n}\right)^2 \times \omega^2} = \sqrt{4\zeta^2} = 2\zeta$$

Hence, this term belongs to denominator.

so, $\text{Magnitude} = 20\log_{10}\left(\frac{1}{2\zeta}\right) = 6$ dB

or, $\log_{10}\left(\frac{1}{2\zeta}\right) = \frac{6}{20}$

Scanned with CamScanner

or, $\frac{1}{2\zeta} = (10)^{0.3}$

or, $\frac{1}{2\zeta} = 2$

$\therefore \zeta = \frac{1}{4} = 0.25$

$\therefore \left[1 + \frac{S^2}{(3)^2} + 2 \times 0.25 \times \frac{S}{3}\right]$; in pole

or, $(0.11S^2 + 0.166\,S + 1)$; in pole

Transfer function of the system $= \frac{C(S)}{R(S)} = \frac{1}{0.11S^2 + 0.166\,S + 1}$

i) Maximum overshoot $(M_p) = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)} = e^{-\left[\frac{\pi \times 0.25}{\sqrt{1-(0.25)^2}}\right]} = 0.44 = 44.43\%$

ii) Peak time $(t_p) = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{3 \times \sqrt{1-(0.25)^2}} = 1.081$ sec.

18. **For a compensator transform function $G_c(S) = \frac{S+\tau}{S+a\tau}$, give the condition of lead compensator. For the given value of a, what is the frequency that leads to maximum phase angle lead? [2071 Chaitra]**

Solution:

Here, transfer function of lead compensator of Bode plot is given by;

$$G_c(S) = \frac{S+\tau}{S+a\tau}$$

i) In lead compensator zero is closer than pole to origin.

For above transfer function;

i.e., $\tau < a\tau$

ii) In lag compensator pole is closer to origin that zero.

i.e., $a\tau < \tau$

Here, transfer function of lead compensator of root locus is;

$$G_c(S) = \frac{S+\tau}{S+a\tau}$$

iii) In lead compensator, zero is closer than pole to origin.

i.e., $\tau < a\tau$

iv) In lag compensator pole is closer to origin.

i.e., $a\tau < \tau$

In phase lead compensation by Bode plot

$$a = \frac{1+\sin\phi_m}{1-\sin\phi_m}$$

where, $\phi_m = (\phi_s - \phi + \epsilon)$

ϵ is the margin of safety.

ϕ_s is the desired P.M.

ϕ is the P.M. of uncompensated system.

and, in lag compensator;

$$\frac{1}{a\tau} = \frac{\text{New g.c.f.}}{10}$$

Scanned with CamScanner

or, $a = \frac{10}{T \times \text{New g.c.f.}}$

NOTE

Lead compensation by Bode plot

The transfer function of compensator is $G_c(S) = \frac{1+aTS}{1+TS}$ and $a > 1$.

i) Bandwidth increases

ii) Gain crossover frequency increases

iii) Zero at $S = -\frac{1}{aT}$

iv) Pole at $S = -\frac{1}{T}$

v) Since, $a > 1$; zero is closer to origin.

vi) With increase in a; increases g.c.f.

Lag compensation by Bode plot

The transfer function of lag compensator is $G_c(S) = \frac{1+aTS}{1+TS}$ and $a < 1$.

i) Bandwidth decreases

ii) Gain crossover frequency decreases

vii) Zero at $S = -\frac{1}{aT}$

iii) Pole at $S = -\frac{1}{T}$

iv) Since, $a < 1$; pole is nearer to origin.

v) With decrease in a; decreases g.c.f.

Lead compensation using root locus

The transfer function of lead compensator is $G_c(S) = \frac{K_C\left(S+\frac{1}{T}\right)}{\left(S+\frac{1}{\alpha T}\right)}$.

i) Bandwidth increases.

ii) Pole at $S = -\frac{1}{\alpha T}$

iii) Zero at $S = -\frac{1}{T}$

iv) $\alpha < 1$

v) Hence, zero is closer to origin.

Lag compensation using root locus

The transfer function of lag compensator is $G_c(S) = \frac{K_C\left(S+\frac{1}{\alpha T}\right)}{\left(S+\frac{1}{T}\right)}$.

i) Bandwidth decreases

ii) $\alpha < 1$; so, pole is closer to origin.

iii) Pole at $S = -\frac{1}{T}$

iv) Zero at $S = -\frac{1}{\alpha T}$

Selection of compensator using Bode plot

i) The lead compensation becomes ineffective when the phase angle of uncompensated system decreases rapidly near the g.c.f. This is due to when two or more poles close to the g.c.f. Under this situation, choose lag compensator.

Scanned with CamScanner

ii) The combination of increased open loop gain and lead compensator improves steady state error and phase margin, due to this noise level increase. Therefore, choose the lag compensator, when reduced noise level required.

iii) In plot, draw Bode plot for uncompensated system. Note the new g.c.f. corresponding to required phase margin. If new g.c.f. is greater than initial g.c.f., choose lag compensator. If not, choose lag compensator.

Selection of compensator using root locus

i) When an improvement in steady state performance, choose a lag compensator.

ii) When an improvement in transient performance is required, lead compensator should be designed.

iii) In lead compensation, you are provided with gain (K) in transfer function. In lag compensation you have to find out value of gain (K).

19. **The open loop transfer function of a unity feedback system is given by $G(S) = \frac{108}{S^2(S+4)(S^2+3S+12)}$. Find the static error coefficients and steady state error of the system when subjected to an input given by $r(t) = 2 + 5t + 8t^2$.** [2074 Ashwin]

Solution:

Determination of K_p, K_v and K_a

See the example 7.5

Here, input $r(t) = 2 + 5t + 8t^2$ consists of step input, ramp input and parabolic input; so, steady state error becomes;

$$e_{ss} = \frac{2}{1+K_p} + \frac{5}{K_v} + \frac{8}{K_a} = \frac{2}{1+\infty} + \frac{5}{\infty} + \frac{8}{\frac{9}{4}} = 0 + 0 + \frac{32}{9} = \frac{32}{9}$$

$\therefore$ $\boxed{e_{ss} = 3.556}$

20. **State whether the statement "Derivative controllers are always used with other controllers" is true or false. Justify you answer.** [2062 Poush]

Solution:

The given statement is true because derivative controllers are always used with other controllers i.e., with either P-controller or with I-controller.

For the further steps

See the definition part 7.6.1.2

Scanned with CamScanner

Chapter 8

STATE SPACE ANALYSIS

8.1 Definition of state–space 286
8.2 State space representation of electrical and mechanical systems 287
8.2.1 State space representation of electrical system 287
8.2.2 State space representation of differential equation 290
8.2.3 State space representation of mechanical system 292
8.3 Conversion from state space to transfer function 293
8.3.1 Stability analysis from state space 296
8.4 Conversion from transfer function (T.F.) to state space 298
8.4.1 Transfer function containing poles only 298
8.4.2 Transfer function containing both poles and zeros 300
8.4.3 Controllability and observality 302
8.5 State transition matrix (STM) 303

8.1 DEFINITION OF STATE–SPACE

State space analysis

The state of a dynamic system is the smallest set of variable (called state variable) such that the knowledge of these variables at $t = t_0$ together with the knowledge of input for $t \geq t_0$ completely determines the behaviour of the system for any time $t > t_0$.

State variables

Set of variables used to describe the state of the network or system.

State vector

If we need n state variable that describes the behaviour of a given system, then n-state variables can be considered as n-component of vector x. Such a vector is called as state vector. Thus, a state vector is that which determines uniquely the system state for any time $t > t_0$.

State space

For n-dimensional space whose coordination axes consists of $x_1, x_2, x_3, \ldots\ldots, x_n$ axis; where, $x_1, x_2, x_3, \ldots\ldots, x_n$ are state variables is called state space. Any state can be represented by a point in the state plane.

State

Minimum amount of information at $t = t_0$ describe the behaviour of the system is called state.

Inductor current and capacitor voltage are state of the network or system.

The notation i_L and V_C are state variables.

Scanned with CamScanner

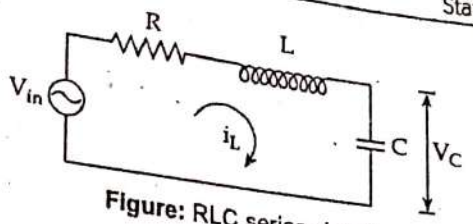


Figure: RLC series circuit

Advantages of state–space

i) This approach can be applied to linear or non–linear, Time variant or time invariant system.
ii) It is time–domain approach.
iii) It is applicable for MIMO (Multiple input multiple output) system.
iv) It is easier to apply where the Laplace transform cannot be applied.
v) n^{th} order differential equation can be expressed as n equation of first order whose solutions are easier.
vi) This method is suitable for digital computer computation because this is a time domain approach.

8.2 STATE SPACE REPRESENTATION OF ELECTRICAL AND MECHANICAL SYSTEMS

In general, the state equation and output equation for any system can be written as;

$\dot{X} = AX + BU$; which is a state equation.

$Y = CX + DU$; which is a state equation.

where, A; B, C, D are transition matrix

U is the input vector.

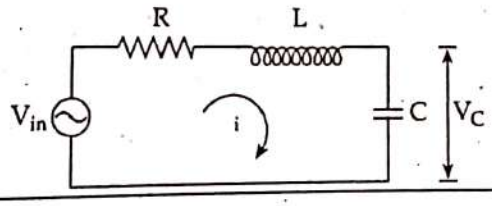
X is the state vector.

$\dot{X} = x_1, x_2, x_3, \ldots\ldots, x_n$ is state variables.

8.2.1 State Space Representation of Electrical System

EXAMPLE 8.1

Write down state space representation of given system.



Solution:

There will be two state variables. One is current through inductor and another is voltage across capacitor *i.e.*, i_L and V_C are state variables.

Now, applying K.V.L.; we have,

$$V_{in} = iR + L\frac{di}{dt} + V_C$$

or, $$\frac{V_{in}}{L} - i\left(\frac{R}{L}\right) - \frac{V_C}{L} = \frac{di}{dt}$$

or, $$\frac{di}{dt} = -\left(\frac{R}{L}\right)i - \left(\frac{1}{L}\right)V_C + \left(\frac{1}{L}\right)V_{in} \quad (1)$$

Also,

$$i = C\frac{dV_C}{dt} \quad (2)$$

or, $$\frac{dV_C}{dt} = \left(\frac{1}{C}\right)i$$

Representing in matrix format; we have,

Scanned with CamScanner

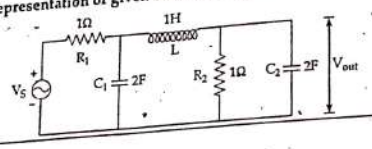
$$\begin{bmatrix}\frac{di}{dt}\\ \frac{dV_C}{dt}\end{bmatrix} = \begin{bmatrix}-\frac{R}{L} & -\frac{1}{L}\\ \frac{1}{C} & 0\end{bmatrix}\begin{bmatrix}i\\ V_C\end{bmatrix} + \begin{bmatrix}\frac{1}{L}\\ 0\end{bmatrix}V_{in}$$; which is in the form of $\dot{X} = AX + BU$.

Considering V_{out} as V_C; we have,

$$V_{out} = [0 \quad 1]\begin{bmatrix}i\\ V_C\end{bmatrix}$$; which is in the form of $Y = CX + DU$.

EXAMPLE 8.2 **[2068 Baishakh]**

Obtain the state space representation of given electrical network.



Solution:

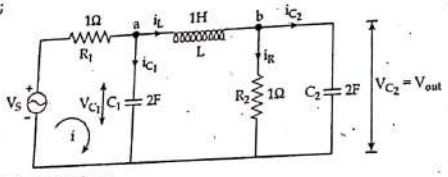
There will be three numbers of state variables.

i) Inductor current (i_L) across 1 H inductor

ii) Capacitor voltage (V_{C_1}) across 2F capacitor

iii) Capacitor voltage (V_{C_2}) across 2F capacitor

The circuit becomes;



Applying K.V.L. in first loop; we have,

$$V_S = iR_1 + V_{C_1} = i + V_{C_1} \quad (1)$$

Applying K.V.L. in second loop; we have,

$$V_S = L\left(\frac{di_L}{dt}\right) + R_2 i_R \quad (2)$$

or, $V_{C_1} = \frac{di_L}{dt} + i_R$

Applying K.V.L. in third loop; we have,

$$V_{C_2} = i_R R_2 = i_R \quad (3)$$

Applying KCL in node a; we have,

$$i = i_{C_1} + i_L = C_1\left(\frac{dV_{C_1}}{dt}\right) + i_L = 2\left(\frac{dV_{C_1}}{dt}\right) + i_L \quad (4)$$

Applying KCL in node b; we have,

$$i_L = i_R + i_{C_2} = i_R + C_2\left(\frac{dV_{C_2}}{dt}\right) = i_R + 2\left(\frac{dV_{C_2}}{dt}\right) \quad (5)$$

Arranging equation (1); we have,

or, $\frac{dV_{C_1}}{dt} = -\left(\frac{1}{2}\right)i_L - \left(\frac{1}{2}\right)V_{C_1} + \frac{V_S}{2}$ [From equation (4)]

Arranging equation (2); we have,

$$V_{C_1} = \frac{di_L}{dt} + i_R \quad (6)$$

or, $V_{C_1} - i_R = \frac{di_L}{dt}$

or, $\frac{di_L}{dt} = V_{C_1} - V_{C_2}$ [$\because V_{C_2} = i_R$]

Arranging equation (3); we have,

$$V_{C_2} = i_R = i_L - 2\left(\frac{dV_{C_2}}{dt}\right) \quad (7)$$

or, $\frac{dV_{C_2}}{dt} = \left(\frac{1}{2}\right)i_L - \left(\frac{1}{2}\right)V_{C_2}$ (8)

Here, we get equation (6), (7) and (8) in the form of inductor current and capacitor voltage only.

Hence, state space representation for input is;

$$\begin{bmatrix}\dot{V}_{C_1}\\ \dot{i}_L\\ \dot{V}_{C_2}\end{bmatrix} = \begin{bmatrix}-\frac{1}{2} & -\frac{1}{2} & 0\\ 1 & 0 & -1\\ 0 & \frac{1}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}V_{C_1}\\ i_L\\ V_{C_2}\end{bmatrix} + \begin{bmatrix}\frac{1}{2}\\ 0\\ 0\end{bmatrix}V_S$$; which is in the form of $\dot{X} = AX + BU$.

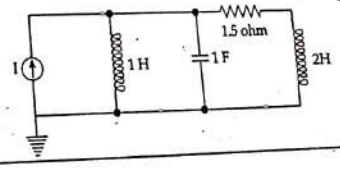
State space representation for output is given by;

$V_{out} = V_{C_2}$

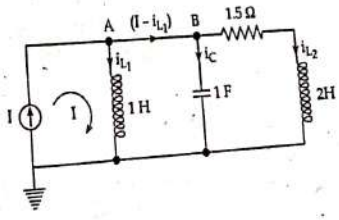
$$\therefore \quad V_{out} = [0 \quad 0 \quad 1]\begin{bmatrix}V_{C_1}\\ i_L\\ V_{C_2}\end{bmatrix}$$; which is in the form of $Y = CX + DU$.

EXAMPLE 8.3

Develop state space equations for the following circuit considering voltage of 1 F capacitor as output. I is the input to the system. **[2072 Chaitra, 2073 Chaitra]**



Solution:



Scanned with CamScanner

Here, three state variables since two inductors and one capacitor. So, in loop (2),

Voltage across 1 H inductor = Voltage across 1 F capacitor

or, $L\left(\frac{di_{L_1}}{dt}\right) = V_C$ (1)

or, $\frac{di_{L_1}}{dt} = V_C$

In loop (3); we have, (2)

$$V_C = i_{L_2} \times R + L \times \frac{di_{L_2}}{dt} = i_{L_2} \times 1.5 + 2\left(\frac{di_{L_2}}{dt}\right) = 1.5i_{L_2} + 2\left(\frac{di_{L_2}}{dt}\right)$$

Applying KCL in node B; we have, (3)

$$I - i_{L_1} = i_{L_2} + i_C \quad (4)$$

or, $I = i_{L_1} + i_{L_2} + i_C$ (5)

and, $V_{out} = V_C$

or, $i_C = C\left(\frac{dV_C}{dt}\right) = \frac{dV_C}{dt}$

Arranging equation (2); we have, (6)

$$\frac{di_{L_2}}{dt} = -\left(\frac{1.5}{2}\right) i_{L_2} + \frac{V_C}{2} = -0.75i_{L_2} + \frac{V_C}{2}$$

Arranging equation (5); we have, (7)

$$\frac{dV_C}{dt} = I - i_{L_1} + i_{L_2}$$

Hence, we got there equations in terms of capacitor voltage and inductor current only.

$$i_{L_1} = V_C$$

$$i_{L_2} = -0.75i_{L_2} + \frac{V_C}{2}$$

$$\dot{V}_C = I - i_{L_1} - i_{L_2}$$

Hence, representing in matrix form; we have,

$$\begin{bmatrix} i_{L_1} \\ i_{L_2} \\ \dot{V}_C \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -0.75 & \frac{1}{2} \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} i_{L_1} \\ i_{L_2} \\ V_C \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} I$$; which is in the form of $\dot{X} = AX + BU$.

Now,

$$\therefore \quad \text{Output } (V_{out}) = V_C = [0 \quad 0 \quad 1] \begin{bmatrix} i_{L_1} \\ i_{L_2} \\ V_C \end{bmatrix}$$; which is in the form of $Y = CX + DU$.

8.2.2 State Space Representation of Differential Equation

EXAMPLE 8.5

Derive the state space model and represent it in the block diagram, whose transfer function is

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 10y = 8\,u(t).$$

Solution:

Here, the order of equation is 3. So, we need three state variables.

Let, $y = x_1$

$$\dot{y} = \dot{x}_1 = \dot{x}_2 = x_3$$

Here, x_1, x_2 and x_3 are state variables. Now, the equation is;

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 10y = 8\,u(t)$$

$$\dddot{y} = 8\,u(t) - 6\ddot{y} - 11\dot{y} - 10y$$

or, $\dot{x}_3 = 8\,u(t) - 6x_3 - 11x_2 - 10x_1$

Now, the state space model is;

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix} u(t)$$

Now, we have to represent it in the block diagram.

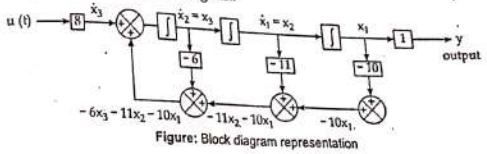


Figure: Block diagram representation

EXAMPLE 8.6

Derive the space model and represent it in the block diagram, whose transfer function is;

$$\frac{d^4y}{dt^4} + \frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + y = u(t)$$

Solution:

The highest order of equation is 4. So, we need four state variables.

Let, $y = x_1$

$$\dot{y} = \dot{x}_1 = x_2$$

$$\ddot{y} = \dot{x}_1 = \dot{x}_2 = x_3$$

$$\dddot{y} = \dot{x}_1 = \dot{x}_2 = \dot{x}_3 = x_4$$

Here, x_1, x_2, x_3 and x_4 are state variables. Now, the question is;

$$\ddddot{y} + \dddot{y} + 4\ddot{y} + 6\dot{y} + y = u(t)$$

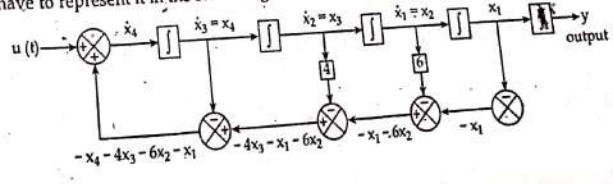
or, $\ddddot{y} = u(t) - \dddot{y} - 4\ddot{y} - 6\dot{y} - y$

or, $\dot{x}_4 = u(t) - x_4 - 4x_3 - 6x_2 - x_1$

The state space model is;

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -6 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

Now, we have to represent it in the block diagram;

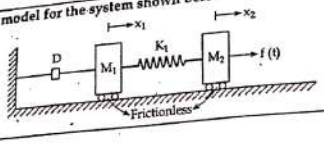


Scanned with CamScanner

8.2.3 State Space Representation of Mechanical System

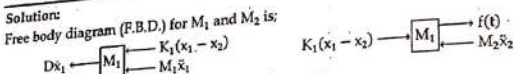
EXAMPLE 8.7

Derive the state variables model for the system shown below:



Solution:

Free body diagram (F.B.D.) for M_1 and M_2 is;



Now, for mass M_2; applying De-Alembert's principle,

$$f(t) + K_1(x_1 - x_2) = M_2\ddot{x}_2 \quad (1)$$

or, $$f(t) = M_2\ddot{x}_2 - K_1(x_1 - x_2)$$

Now, for mass M_1; applying De-Alembert's principle,

$$K_1(x_1 - x_2) + M_1\ddot{x}_1 + D\dot{x}_1 = 0 \quad (2)$$

The order of equation is 2; we have,

$2 \times 2 = 4$ state variables [2 in each equation]

For first equation;

Let, $x_1 = z_1$

or, $\dot{x}_1 = \dot{z}_1 = z_2$

and, $x_2 = z_3$

or, $\dot{x}_2 = \dot{z}_3 = z_4$

So, equation (1) becomes;

$$f(t) = M_2\ddot{x}_2 - K_1(x_1 - x_2)$$

$$f(t) = M_2\dot{z}_4 - K_1x_1 - K_1x_2$$

$$f(t) = M_2\dot{z}_4 + Kz_1 - Kz_3$$

so, $$\dot{z}_4 = \left(\frac{1}{M_2}\right)f(t) + \left(\frac{K_1}{M_2}\right)z_1 - \left(\frac{K_1}{M_2}\right)z_3 \quad (3)$$

and equation (2) becomes;

$$K_1x_1 - K_1x_2 + M_1\ddot{x}_1 + D\dot{x}_1 = 0$$

or, $$M_1\ddot{x}_1 = -K_1x_1 + K_1x_2 - D\dot{x}_1$$

or, $$M_1\dot{z}_2 = -K_1z_1 + K_1z_3 - Dz_2$$

or, $$\dot{z}_2 = -\left(\frac{K_1}{M_1}\right)z_1 + \left(\frac{K_1}{M_1}\right)z_3 - \left(\frac{D}{M_1}\right)z_2 \quad (4)$$

So, the state space model becomes;

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\\ \dot{z}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ -\left(\frac{K_1}{M_1}\right) & -\left(\frac{D}{M_1}\right) & \left(\frac{K_1}{M_1}\right) & 0\\ 0 & 0 & 0 & 1\\ \left(\frac{K_1}{M_2}\right) & 0 & -\left(\frac{K_1}{M_2}\right) & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\\ z_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ \frac{1}{M_2}\end{bmatrix}f(t)$$

x_1 and x_2 are displacements and hence output of the mechanical system.

i.e., $x_1 = z_1$

and, $x_2 = z_3$

so, $$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}z_1\\ z_3\end{bmatrix}$$

8.3 CONVERSION FROM STATE SPACE TO TRANSFER FUNCTION

We have, the input equation is;

$$\dot{X} = AX + BU \quad (1)$$

and, output equation is;

$$Y = CX + DU \quad (2)$$

where, X is the state variable.

U is the input.

Y is the output.

A, B, C, D is the state transition matrix.

Taking Laplace transform for both (1) and (2) equation;

$$SX(S) = AX(S) + BU(S)$$

or, $$SIX(S) = AX(S) + BU(S)$$

where, I is the identity matrix of order same as that of A.

$$[SI - A]X(S) = BU(S)$$

or, $$X(S) = B[SI - A]^{-1}U(S) \quad (3)$$

For equation (2); we have,

$$Y(S) = CX(S) + DU(S)$$

Substituting the value of the equation (3); we get,

$$Y(S) = C[B(SI - A)^{-1}]U(S) + DU(S) = [C(SI - A)^{-1}B + D]U(S)$$

$$\therefore \quad \boxed{\frac{Y(S)}{U(S)} = C[SI - A]^{-1}B + D} \quad (4)$$

which is required transfer function of the system for SISO (single input single output) system 'C' and written as C^T.

i.e., $$\boxed{\text{T.F.} = G(S) = \frac{Y(S)}{U(S)} = C^T[SI - A]^{-1}B + D}$$

EXAMPLE 8.9

The system equations for state model are given by $\dot{x}(t) = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}x(t) + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)$ and $y(t) = [1 \quad 0]x(t)$. Find the transfer function of the given system.

Solution:

We know that;

$$\dot{x}(t) = AX + BU$$

and, $$y(t) = CX + DU$$

Comparing, $A = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$, $B = \begin{bmatrix}0\\ 1\end{bmatrix}$, $C = C^T = [1 \quad 0]$ and $D = [0]$

We know that,

$$\text{T.F.} = G(S) = C^T[SI - A]^{-1}B + D$$

Scanned with CamScanner

Now, $[SI - A] = S\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix} = \begin{bmatrix}S & 0\\ 0 & S\end{bmatrix} - \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix} = \begin{bmatrix}S & -1\\ 2 & S+3\end{bmatrix}$

Again, $[SI - A]^{-1} = \frac{\text{Adjoint of } [SI - A]}{|SI - A|} = \frac{1}{S^2 + 3S + 2} \times \begin{bmatrix}S+3 & 1\\ -2 & S\end{bmatrix}$

and, $T.F. = G(S) = C^T[SI - A]^{-1}B + D = [1 \quad 0] \times \frac{1}{S^2 + 3S + 2} \times \begin{bmatrix}S+3 & 1\\ -2 & S\end{bmatrix} \times \begin{bmatrix}0\\ 1\end{bmatrix} + [0]$

$= \frac{1}{S^2 + 3S + 2} \times [1 \quad 0] \times \begin{bmatrix}1\\ S\end{bmatrix} = \frac{1}{S^2 + 3S + 2} \times 1 = \frac{1}{S^2 + 3S + 2}$

Hence, $T.F. = \frac{1}{S^2 + 3S + 2} = \frac{1}{(S+1)(S+2)}$

$\therefore \boxed{T.F. = G(S) = \frac{1}{(S+1)(S+2)} = \frac{Y(S)}{U(S)}}$

EXAMPLE 8.10

Given state equation and output equation, determine the poles and zeros of given system.

$\dot{x}(t) = \begin{bmatrix}0 & 1 & 0\\ -1 & -1 & 0\\ 1 & 0 & 0\end{bmatrix} X + \begin{bmatrix}0\\ 1\\ 0\end{bmatrix} u$ and $Y = [0 \quad 0 \quad 1]X$ [2072 Kartik, 2074 Ashwin]

Solution:

We know that;

$\dot{x}(t) = AX(t) + BU(t)$; input equation

and, $y(t) = CX(t) + DU(t)$; output equation

Comparing $A = \begin{bmatrix}0 & 1 & 0\\ -1 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}$, $B = \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$, $C = C^T = [0 \quad 0 \quad 1]$ and $D = [0]$

Now, I is the identity matrix having same order as that of A.

$[SI - A] = S\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1 & 0\\ -1 & -1 & 0\\ 1 & 0 & 0\end{bmatrix} = \begin{bmatrix}S & 0 & 0\\ 0 & S & 0\\ 0 & 0 & S\end{bmatrix} - \begin{bmatrix}0 & 1 & 0\\ -1 & -1 & 0\\ 1 & 0 & 0\end{bmatrix} = \begin{bmatrix}S & -1 & 0\\ 1 & S+1 & 0\\ -1 & 0 & S\end{bmatrix}$

$[SI - A] = S\begin{vmatrix}S+1 & 0\\ 0 & S\end{vmatrix} - 1\begin{vmatrix}-1 & 0\\ 0 & S\end{vmatrix} + (-1)\begin{vmatrix}-1 & 0\\ S+1 & 0\end{vmatrix} = S^2(S+1) - 1[-S] = S^3 + S^2 + S$

and, $[SI - A]^{-1} = \frac{\text{Adjoint of } |SI - A|}{|SI - A|} = \frac{\text{Transpose of matrix of cofactor}}{|SI - A|}$

$= \frac{1}{S^3 + S^2 + S} \times \begin{bmatrix}S(S+1) & S & 0\\ -S & S^2 & 0\\ (S+1) & 1 & S^2 + S + 1\end{bmatrix}$

Now,

$T.F. = \frac{Y(S)}{U(S)} = G(S) = C^T[SI - A]^{-1}B + D$

$= [0 \quad 0 \quad 1] \times \frac{1}{S(S^2 + S + 1)} \times \begin{bmatrix}S^2 + S & S & 0\\ -S & S^2 & 0\\ (S+1) & 1 & S^2 + S + 1\end{bmatrix}\begin{bmatrix}0\\ 1\\ 0\end{bmatrix} + [0]$

$= [0 \quad 0 \quad 1] \times \frac{1}{S(S^2 + S + 1)} \times \begin{bmatrix}S\\ S^2\\ 1\end{bmatrix} = \frac{1}{S(S^2 + S + 1)} \times 1 = \frac{1}{S(S^2 + S + 1)}$

$\therefore \boxed{T.F. = \frac{Y(S)}{U(S)} = G(S) = \frac{1}{S(S^2 + S + 1)}}$

Pole of the system at $S_1 = 0, S_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}j, S_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}j$

No zero of the system.

EXAMPLE 8.11

Find transfer function of the following system:

$\dot{X} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -3\end{bmatrix} X + \begin{bmatrix}10\\ 0\\ 0\end{bmatrix} u$ and $Y = [1 \quad 0 \quad 0]X$. [2071 Shrawan]

Solution:

Comparing with;

$\dot{X} = AX + BU$; for input equation

$Y = CX + DU$; for output equation

so, $A = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -3\end{bmatrix}$, $B = \begin{bmatrix}10\\ 0\\ 0\end{bmatrix}$, $C = C^T = [1 \quad 0 \quad 0]$ and $D = [0]$

Now,

$T.F. = \frac{Y(S)}{U(S)} = G(S) = C^T[SI - A]^{-1}B + D$

We have to find $[SI - A]$ first;

$[SI - A] = S\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -3\end{bmatrix} = \begin{bmatrix}S & -1 & 0\\ 0 & S & -1\\ 1 & 2 & S+3\end{bmatrix}$

Now,

Cofactor of $a_{11} = \begin{vmatrix}S & -1\\ 2 & S+3\end{vmatrix} = S(S+3) + 2 = S^2 + 3S + 2$

Cofactor of $a_{12} = -\begin{vmatrix}0 & -1\\ 1 & S+3\end{vmatrix} = -1$

Cofactor of $a_{13} = \begin{vmatrix}0 & S\\ 1 & 2\end{vmatrix} = -S$

Cofactor of $a_{21} = -\begin{vmatrix}-1 & 0\\ 2 & S+3\end{vmatrix} = (S+3)$

Cofactor of $a_{22} = \begin{vmatrix}S & 0\\ 1 & S+3\end{vmatrix} = S(S+3)$

Cofactor of $a_{23} = -\begin{vmatrix}S & -1\\ 1 & 2\end{vmatrix} = -(2S+1)$

Cofactor of $a_{31} = \begin{vmatrix}-1 & 0\\ S & -1\end{vmatrix} = 1$

Cofactor of $a_{32} = -\begin{vmatrix}S & 0\\ 0 & -1\end{vmatrix} = S$

Scanned with CamScanner

Cofactor of $a_{33} = \begin{vmatrix} S & -1 \\ 0 & S \end{vmatrix} = S^2$

Matrix of cofactor $= \begin{bmatrix} S^2+3S+2 & -1 & -S \\ S+3 & S(S+3) & -(2S+1) \\ 1 & S & S^2 \end{bmatrix}$

Adjoint of $[SI-A] = \begin{bmatrix} S^2+3S+2 & S+3 & 1 \\ -1 & S(S+3) & S \\ -S & -(2S+1) & S^2 \end{bmatrix}$

Now,

$$|SI-A| = S\begin{vmatrix} S & -1 \\ 2 & S+3 \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ 1 & S+3 \end{vmatrix} + 0 = S[S(S+3)+2] + 1[1] = S[S^2+3S+2]$$

Again,

$$[SI-A]^{-1} = \frac{\text{Adjoint of } |SI-A|}{|SI-A|} = \frac{1}{S(S^2+3S+2)} \times \begin{bmatrix} S^2+3S+2 & S+3 & 1 \\ -1 & S(S+3) & S \\ -S & -(2S+1) & S^2 \end{bmatrix}$$

and, $$T.F. = \frac{Y(S)}{U(S)} = G(S) = C^T[SI-A]^{-1}B + D$$

$$= [1 \quad 0 \quad 0] \times \frac{1}{S(S^2+3S+2)} \times \begin{bmatrix} S^2+3S+2 & S+3 & 1 \\ -1 & S(S+3) & S \\ -S & -(2S+1) & S^2 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$= [1 \quad 0 \quad 0] \times \frac{1}{S(S^2+3S+2)} \times \begin{bmatrix} 10(S^2+3S+2) \\ -10 \\ -10S \end{bmatrix}$$

$$= 10(S^2+3S+2) \times \frac{1}{S(S^2+3S+2)} = \frac{10}{S}$$

$$\therefore \boxed{T.F. = \frac{Y(S)}{U(S)} = G(S) = \frac{10}{S}}$$

8.3.1 Stability Analysis From State Space

We know that;

$$T.F. = G(S) = C^T[SI-A]^{-1}B + D$$

and, $$[SI-A]^{-1} = \frac{\text{Adjoint of } |SI-A|}{\det |SI-A|}$$

so, $$T.F. = G(S) = \frac{C^T \times \text{Adjoint of } |SI-A|B}{|SI-A|} + D$$

$$= \frac{C^T \times \text{Adjoint of } |SI-A|B + D|SI-A|}{|SI-A|}$$

The characteristics equation is given by a denominator term hence, $|SI-A| = 0$ is characteristics equation.

On solving; we have,

If, S = L.H.P., stable
= R.H.P., unstable
= Single pole on imaginary axis, marginally stable
= Multiple pole on imaginary axis; unstable

EXAMPLE 8.12

Obtain characteristics equation for the system having given state model.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u \text{ and } Y = [1 \quad 2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

[2070 Chaitra]

Solution:

We know that;

Transfer function $= G(S) = C^T[SI-A]^{-1}B + D$

Form question comparing with;

$\dot{X} = AX + BU$; for input equation

and, $Y = CX + DU$; for output equation

We get,

$$A = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, C = C^T = [1 \quad 2] \text{ and } D = [0]$$

Now,

$$T.F. = G(S) = C^T[SI-A]^{-1}B + D = \frac{C^T \times \text{Adjoint of } |SI-A|B}{|SI-A|} + D$$

$$= \frac{C^T \times \text{Adjoint of } |SI-A|B + D|SI-A|}{|SI-A|}$$

The denominator term when equation with zero, it gives characteristics equation. Hence,

$$|SI-A| = 0$$

so, $$[SI-A] = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} S+5 & 1 \\ -3 & S+1 \end{bmatrix}$$

Now,

$$\det[SI-A] = |SI-A| = (S+1)(S+5)+3 = S^2+6S+8$$

Hence, the characteristics equation is;

$$|SI-A| = 0$$

or, $$S^2+6S+8 = 0$$

or, $$(S+4)(S+2) = 0$$

Either, $S = -4$ or, $S = -2$

Hence, pole of the system lies in $S_1 = -4$ and $S_2 = -2$ i.e., in the left half S-plane, so the system is stable.

EXAMPLE 8.13

The system equation is given by $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$ and $y(t) = [1 \quad 0] x(t)$. Find transfer function of the system and also check stability. **[2066 Chaitra, 2073 Shrawan]**

Solution:

The input equation for state model is;

$\dot{X} = AX + BU$

and, output equation for state model is;

$Y = CX + DU$

Comparing; $A = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = C^T = [1 \quad 0]$ and $D = [0]$; we get,

Transfer function (T.F.) $= G(S) = \frac{Y(S)}{U(S)} = C^T[SI-A]^{-1}B + D$

Scanned with CamScanner

First of all we have to find $[SI - A]$;

$$[SI - A] = S\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1\\ -5 & -6\end{bmatrix} = \begin{bmatrix}S & -1\\ 5 & S+6\end{bmatrix}$$

$$|SI - A| = S(S+6) - (-5) = S^2 + 6S + 5$$

so, $$[SI - A]^{-1} = \frac{\text{Adjoint of } |SI - A|}{|SI - A|} = \frac{1}{S^2 + 6S + 5}\begin{bmatrix}S+6 & 1\\ -5 & S\end{bmatrix}$$

Now,

$$\text{Transfer function (T.F.)} = C^T[SI - A]^{-1}B + D = [1 \quad 0] \times \frac{1}{S^2 + 6S + 5} \times \begin{bmatrix}S+6 & 1\\ -5 & S\end{bmatrix} \times \begin{bmatrix}0\\ 1\end{bmatrix} + [0]$$

$$= [1 \quad 0] \times \frac{1}{S^2 + 6S + 5} \times \begin{bmatrix}1\\ S\end{bmatrix} = 1 \times \frac{1}{S^2 + 6S + 5} = \frac{1}{S^2 + 6S + 5}$$

Now, characteristics equation is;

$$|SI - A| = 0$$

or, $$S^2 + 6S + 5 = 0$$

or, $$(S+1)(S+5) = 0$$

Either, $S = -1$ or, $S = -5$

The poles of the system lies at $S_1 = -1$ and $S_1 = -5$ *i.e.*, in the left half of S-plane; so, the system is stable.

8.4 CONVERSION FROM TRANSFER FUNCTION (TF) TO STATE SPACE

8.4.1 Transfer Function Containing Poles Only

EXAMPLE 8.15

Derive the state space model for the system below whose Transfer function is given by;

$$\frac{Y(S)}{U(S)} = \frac{K}{S^3 + a_3S^2 + a_2S + a_1}$$

Solution:

Here,

$$\text{T.F.} = \frac{Y(S)}{U(S)} = \frac{K}{S^3 + a_3S^2 + a_2S + a_1}$$

$$S^3Y(S) + a_3S^2Y(S) + a_2SY(S) + a_1Y(S) = K \cdot U(S)$$

Taking inverse Laplace transform; we have,

or, $$\dddot{y}(t) + a_3\ddot{y}(t) + a_2\dot{y}(t) + a_1y(t) = K \cdot U(t)$$

Let, $y = x_1$

$$\dot{y} = \dot{x}_1 = x_2$$

$$\ddot{y} = \ddot{x}_1 = \dot{x}_2 = x_3$$

and, the equation becomes;

$$\dddot{y}(t) = K \cdot U(t) - a_1y(t) - a_2\dot{y}(t) - a_3\ddot{y}(t)$$

or, $$\dot{x}_3 = K \cdot U(t) - a_1x_1 - a_2x_2 - a_3x_3$$

We have three state variables since the order of equation is 3.

The x_1, x_2 and x_3 are state variables. So, the state space model becomes;

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_1 & -a_2 & -a_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ K\end{bmatrix}u(t)$$; which is in the form of $\dot{X} = AX + BU$ the input equation.

Scanned with CamScanner

Now, the output equation is;

$y = x_1$; which is the form of $Y = CX + DU$;

so, $y = [1 \quad 0 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Hence, these are the state representation of transfer function.

EXAMPLE 8.16

A system has transfer function $\frac{Y(S)}{U(S)} = \frac{2}{S^3 + 6S^2 + 11S + 6}$. Find the state and output equation in matrix form and test the controllability and observality of the system. [2069 Chaitra]

Solution:

Here,

$$T.F. = G(S) = \frac{Y(S)}{U(S)} = \frac{2}{S^3 + 6S^2 + 11S + 6}$$

or, $S^3 \cdot Y(S) + 6S^2 \cdot Y(S) + 11S \cdot Y(S) + 6 \cdot Y(S) = 2 \cdot U(S)$

Taking inverse Laplace transform; we get,

$$\dddot{y}(t) + 6\ddot{y}(t) + 11\dot{y}(t) + 6y(t) = 2 \cdot U(t)$$

or, $\dddot{y}(t) = 2 \cdot U(t) - 6y(t) - 11\dot{y}(t) - 6\ddot{y}(t)$

Let, $y = x_1$

$\dot{y} = \dot{x}_1 = x_2$

$\ddot{y} = \ddot{x}_1 = \dot{x}_2 = x_3$

The order of differential equation is three, so there are three state variables x_1, x_2 and x_3.

The state space model becomes;

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(t)$$

Since, $\dddot{y} = 2 \cdot U(t) - 6y - 11\dot{y} - 6\ddot{y}$

$\dot{x}_3 = 2 \cdot U(t) - 6x_1 - 11x_2 - 6x_3$

which is in the form of $\dot{X} = AX + BU$ and the output equation is;

$y = x_1$

or, $y = [1 \quad 0 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; which is in the form of $Y = CX + DU$.

These are the state space representation of transfer function.

For controllability and observality; we have,

$\dot{X} = AX + BU$

and, $Y = CX + DU$

Comparing, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, $C = [1 \quad 0 \quad 0]$ and $D = [0]$; we get,

For controllability,

$Q = [B : AB : A^2B]$ is equal to rank n.

Scanned with CamScanner

For observability,

$Q' = [C^T : A^TC^T : (A^T)^2C^T]$ is equal to rank n.

Now,

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -12 \end{bmatrix}$$

$$A^2 = A \times A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & -11 & -6 \\ 36 & 60 & 25 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ -6 & -11 & -6 \\ 36 & 60 & 25 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -12 \\ 50 \end{bmatrix}$$

so, $Q = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & -12 \\ 2 & -12 & 50 \end{bmatrix}$

Here, no any or column is zero and hence, rank of Q is 3. It is full of rank hence, the system is controllable.

For observality;

$$C^T = \text{Transpose of matrix } C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A^TC^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$(A^T)^2 = A^T \times A^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} = \begin{bmatrix} 0 & -6 & 36 \\ 0 & -11 & 60 \\ 1 & -6 & 25 \end{bmatrix}$$

$$(A^T)^2C^T = \begin{bmatrix} 0 & -6 & 36 \\ 0 & -11 & 60 \\ 1 & -6 & 25 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad Q' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Here, no any row or column is zero and hence rank of Q is three. It is full of rank hence, the system is observable. Hence, the system is controllability and observable.

8.4.2 Transfer Function Containing both Poles and Zeros

EXAMPLE 8.17

A system has transfer function $\frac{Y(S)}{U(S)} = \frac{K(C_2S + C_1)}{S^3 + a_3S^2 + a_2S + a_1}$. Find the state and output equation in matrix form.

Solution:

Transfer function contains zero; so,

$$\frac{Y(S)}{U(S)} = \frac{X_1(S)}{U(S)} \times \frac{Y(S)}{X_1(S)} = \frac{K}{S^3 + a_3S^2 + a_2S + a_1}(C_2S + C_1)$$

Let, $\frac{X_1(S)}{U(S)} = \frac{K}{S^3 + a_3S^2 + a_2S + a_1}$

and, $\frac{Y(S)}{X_1(S)} = (C_2S + C_1)$ (1)

Taking inverse Laplace transform of equation (1); we have,

$S^3 + a_3S^2 + a_2S + a_1X_1(S) = K \cdot U(S)$ (2)

We get,

$\dddot{x}_1(t) + a_3\ddot{x}_1(t) + a_2\dot{x}_1(t) + a_1x(t) = K \cdot U(t)$

Let, $x_1 = z_1$

$\dot{x}_1 = \dot{z}_1 = z_2$

$\ddot{x}_1 = \ddot{z}_1 = \dot{z}_2 = z_3$

and, $\dddot{x}_1 = K \cdot U(t) - a_1x(t) - a_2\dot{x}_1(t) - a_3\ddot{x}_1(t) = K \cdot U(t) - a_1z_1 - a_2z_2 - a_3z_3$

so, $\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix}u(t)$; which is in the form of $\dot{z} = AZ + BU$.

Taking inverse Laplace transfer of equation (2); the output equation becomes,

$Y(S) = (C_2S + C_1)X_1(S)$

We get,

$y(t) = C_2\dot{x}_1(t) + C_1x_1(t) = C_2z_2 + C_1z_1$

so, $y(t) = [C_1 \quad C_2 \quad 0]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$; which is output and in the form of $y = CZ + DU$.

EXAMPLE 8.18

A system has transfer function $G(S) = \frac{20(10S + 1)}{S^3 + 3S^2 + 2S + 1}$.

Find the state and output equation in matrix form.

Solution:

Transfer function contains zero.

so, $G(S) = \frac{Y(S)}{U(S)} = \frac{20}{S^3 + 3S^2 + 2S + 1} \times (10S + 1)$ (1)

Let, $\frac{Y(S)}{U(S)} = \frac{X_1(S)}{U(S)} \times \frac{Y(S)}{X_1(S)}$ (2)

Now, comparing;

$$\frac{X_1(S)}{U(S)} = \frac{20}{S^3 + 3S^2 + 2S + 1}$$

or, $[S^3 + 3S^2 + 2S + 1]X_1(S) = 20 \cdot U(S)$

or, $S^3X_1(S) + 3S^2X_1(S) + 2SX_1(S) + X_1(S) = 20 \cdot U(S)$

Taking inverse Laplace transform; we have,

$\dddot{x}_1(t) + 3\ddot{x}_1(t) + 2\dot{x}_1(t) + x_1(t) = 20 \cdot U(t)$

or, $\dddot{x}_1(t) = 20 \cdot U(t) - x_1(t) - 2\dot{x}_1(t) - 3\ddot{x}_1(t)$

Let, $x_1(t) = z_1$

Scanned with CamScanner

$\dot{x}_1(t) = \dot{z}_1 = z_2$

$\ddot{x}_1(t) = \dot{z}_1 = \dot{z}_2 = z_3$

$\dddot{x}_1(t) = 20 \cdot U(t) - z_1 - 2z_2 - 3z_3$

Now, the state matrix for input equation is;

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -3\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 20\end{bmatrix}u(t)$$; which is in the form of $\dot{z} = AZ + BU$.

From the equation, comparing; we get,

$$\frac{Y(S)}{X_1(S)} = (10S + 1)$$

or, $Y(S) = (10S + 1)X_1(S)$

Taking inverse Laplace transform; we have,

$y(t) = 10\dot{x}_1(t) + x_1(t) = 10z_2 + z_1$

So, the state space model for output equation becomes;

$$y(t) = [1 \quad 10 \quad 0]\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}$$; which is in the form of $y = CZ + DU$.

8.4.3 Controllability and Observability

This is given by Kalman and also sometimes called Kalman's test.

Let, $\dot{X} = AX + BU$; input equation

$Y = CX + DU$; output equation

A linear time invariant is completely controllable if and only if the rank of the controllability matrix is defined as is equal to rank n.

$Q = [B : AB : A^2B : \ldots\ldots A^{n-1}B]$

The system is observable if and only if the rank of the observality matrix is defined as is equal to rank n.

$Q' = [C^T : A^TC^T : (A^T)^2C^T : \ldots\ldots (A^T)^{n-1}C^T]$

where, C^T = Transpose of matrix C.

A^T = Transpose of matrix A.

EXAMPLE 8.19

Consider the following system. $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-0.5 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)$ **and** $y(t) = [0 \quad 1]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$. **Test for controllability and observality.**

Solution:

Comparing with input equation $\dot{X} = AX + BU$ and with output equation $Y = CX + DU$; we get;

$A = \begin{bmatrix}-0.5 & 0\\ 0 & -2\end{bmatrix}$, $B = \begin{bmatrix}0\\ 1\end{bmatrix}$, $C = [0 \quad 1]$ and $D = [0]$

Now, to test controllability the matrix is $Q[B : AB]$

so, $AB = \begin{bmatrix}-0.5 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}0\\ -2\end{bmatrix}$

so, matrix Q is;

$Q[B : AB] = \begin{bmatrix}0 & 0\\ 1 & -2\end{bmatrix}$

Rank of Q is one. The rank should be two hence, system is **uncontrollable.**

Now, to test observality the matrix is $Q'[C^T : A^TC^T]$

$C^T = \begin{bmatrix}0\\ 1\end{bmatrix}$

$A^TC^T = \begin{bmatrix}-0.5 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}0\\ -2\end{bmatrix}$

so, Matrix $Q' = \begin{bmatrix}0 & 0\\ 1 & -2\end{bmatrix}$

Rank of Q' is one. The rank should be two hence system is **unobservable.**

Hence, given system is uncontrollable and unobservable.

8.5 STATE TRANSITION MATRIX (STM)

For unforced input *i.e.*, $U(t) = 0$

The state of any system is determined by a matrix called state transition matrix.

We know that;

$\dot{X}(t) = Ax(t) + Bu(t)$

$u(t) = 0$ gives;

$\dot{X}(t) = Ax(t)$

Taking Laplace transform; we have,

$SX(S) - X(O) = AX(S)$

or, $[SI - A]X(S) = X(O)$ (1)

or, $X(S) = [SI - A]^{-1}X(O)$

Let, $\phi(S) = [SI - A]^{-1}$ = Resolvent matrix;

Taking inverse Laplace transform; we have,

$\phi(t) = L^{-1}[\phi(S)] = L^{-1}[(SI - A)^{-1}]$

Then,

$\phi(t) = e^{At}$ = State transition matrix (STM)

EXAMPLE 8.20

Evaluate state transition matrix of system represented by $\frac{d}{dt}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}-5 & -6\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u$. **[2070 Ashadh**

Solution:

Here,

$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-5 & -6\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u$

Comparing with input equation $\dot{X} = AX + BU$; we get,

$A = \begin{bmatrix}-5 & -6\\ 1 & 0\end{bmatrix}$

and, $B = \begin{bmatrix}1\\ 0\end{bmatrix}$

Now,

$$[SI - A] = S\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}-5 & -6\\ 1 & 0\end{bmatrix} = \begin{bmatrix}S & 0\\ 0 & S\end{bmatrix} - \begin{bmatrix}-5 & -6\\ 1 & 0\end{bmatrix} = \begin{bmatrix}S+5 & 6\\ -1 & S\end{bmatrix}$$

$\det [SI - A] = |SI - A| = S(S + 5) + 6 = S^2 + 5S + 6$

Again,

Scanned with CamScanner

$$[SI - A]^{-1} = \frac{\text{Adjoint of } [SI - A]}{|SI - A|} = \frac{1}{S^2 + 5S + 6}\begin{bmatrix} S & -6 \\ 1 & S+5 \end{bmatrix} = \begin{bmatrix} \frac{S}{(S+3)(S+2)} & \frac{-6}{(S+2)(S+3)} \\ \frac{1}{(S+2)(S+3)} & \frac{(S+5)}{(S+2)(S+3)} \end{bmatrix}$$

We know that,

State transition matrix (STM) $= e^{At} = L^{-1}[SI - A]^{-1}$

Now,

$$L^{-1}[a_{11}] = L^{-1}\left[\frac{S}{(S+2)(S+3)}\right] = L^{-1}\left[\frac{3}{(S+3)} - \frac{2}{(S+2)}\right] = 3e^{-3t} - 2e^{-2t}$$

$$L^{-1}[a_{12}] = L^{-1}\left[\frac{-6}{(S+2)(S+3)}\right] = -6L^{-1}\left[\frac{1}{(S+2)(S+3)}\right] = -6L^{-1}\left[\frac{1}{(S+2)} - \frac{1}{(S+3)}\right]$$

$$= -6e^{-2t} + 6e^{-3t}$$

$$L^{-1}[a_{21}] = L^{-1}\left[\frac{1}{(S+2)(S+3)}\right] = L^{-1}\left[\frac{1}{(S+2)} - \frac{1}{(S+3)}\right] = e^{-2t} - e^{-3t}$$

and, $L^{-1}[a_{22}] = L^{-1}\left[\frac{S+5}{(S+2)(S+3)}\right] = L^{-1}\left[\frac{3}{(S+2)} - \frac{2}{(S+3)}\right] = 3e^{-2t} - 2e^{-3t}$

$\therefore$ State transition matrix (STM) $= e^{At} = \begin{bmatrix} 3e^{-3t} - 2e^{-2t} & -6e^{-2t} + 6e^{-3t} \\ e^{-2t} - e^{-3t} & 3e^{-2t} - 2e^{-3t} \end{bmatrix}$

EXAMPLE 8.21

Evaluate the state transition matrix of system represented by $\dot{x}_1 = -x_1 + x_2$, $\dot{x}_2 = -2x_2$.

Solution:

The state space model becomes;

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Comparing with $\dot{X} = AX + BU$ for input; we get,

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$[SI - A] = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} S+1 & -1 \\ 0 & S+2 \end{bmatrix}$$

$$|SI - A| = (S+1)(S+2) = S^2 + 3S + 2$$

Now,

$$[SI - A]^{-1} = \frac{\text{Adjoint of } [SI - A]}{|SI - A|} = \frac{1}{S^2 + 3S + 2}\begin{bmatrix} S+2 & 1 \\ 0 & S+1 \end{bmatrix}$$

$$= \frac{1}{(S+1)(S+2)}\begin{bmatrix} S+2 & 1 \\ 0 & S+1 \end{bmatrix} = \begin{bmatrix} \frac{1}{(S+1)} & \frac{1}{(S+1)(S+2)} \\ 0 & \frac{1}{(S+2)} \end{bmatrix}$$

Again,

State transition matrix (STM) $= L^{-1}[SI - A]^{-1} = e^{At}$

Scanned with CamScanner

$$\therefore \quad L^{-1}[a_{11}] = L^{-1}\left[\frac{1}{(S+1)}\right] = e^{-t}$$

$$L^{-1}[a_{12}] = L^{-1}\left[\frac{1}{(S+1)(S+2)}\right] = L^{-1}\left[\frac{1}{(S+1)} - \frac{1}{(S+2)}\right] = e^{-t} - e^{-2t}$$

$$L^{-1}[a_{21}] = L^{-1}[0] = 0$$

and, $$L^{-1}[a_{22}] = L^{-1}\left[\frac{1}{(S+2)}\right] = e^{-2t}$$

$$\therefore \quad (STM) = L^{-1}[SI - A]^{-1} = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

Scanned with CamScanner

8.6 SOLVED PROBLEMS

1. Check the stability for the following state space model $A = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, $B = \begin{bmatrix}2\\ 3\end{bmatrix}$, $C = [1 \quad 0]$ and $D = [0]$. [2061 Baishakh]

Solution:

We know characteristics equation of the system is;

$$|SI - A| = 0$$

or, $$[SI - A] = S\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = \begin{bmatrix}S-1 & -2\\ -3 & S-4\end{bmatrix}$$

Now,

$$\det[SI - A] = |SI - A| = (S-1)(S-4) - 6 = S^2 - 5S - 2$$

so, the characteristics equation is;

$$S^2 - 5S - 2 = 0$$

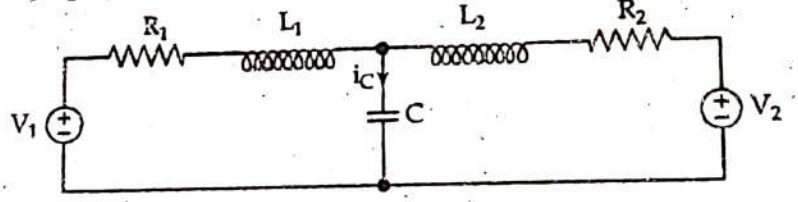
Solving; we get,

$$S_1 = 5.3723$$

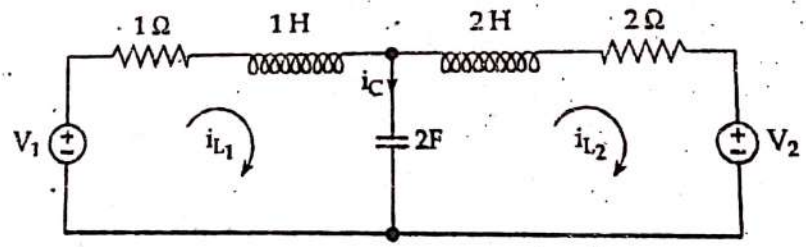
and, $$S_2 = -0.3723$$

The poles of the characteristics lie on $S_1 = 5.3723$ and $S_2 = -0.3723$ *i.e.*, one lies on LH.P. and another on R.H.P. Since, one root lies on R.H.P., the system is unstable.

2. Find state space representation of the network below. Consider current through capacitor (i_C) as output. C = 2F, $R_1 = 1$ ohm, $R_2 = 2$ ohm, $L_1 = 1$ H and $L_2 = 2$ H [2066 Magh]



Solution:



Here, two inductors and one capacitor so three state variables (i_{L_1}, i_{L_2} and V_C).

In loop (1); we have,

$$V_1 = i_{L_1} \times 1 + 1 \times \frac{di_{L_1}}{dt} = i_{L_1} + \frac{di_{L_1}}{dt} \quad (1)$$

In loop (2); we have,

$$V_C = i_{L_2} \times 2 + 2 \times \frac{di_{L_2}}{dt} + V_2 = 2i_{L_2} + 2\left(\frac{di_{L_2}}{dt}\right) + V_2 \quad (2)$$

Again,

$$C\left(\frac{dV_C}{dt}\right) = i_C$$

or, $$\frac{dV_C}{dt} = \frac{i_C}{2} \quad (3)$$

Scanned with CamScanner

Applying K.C.L.; we have,

$$i_{L_1} = i_C + i_{L_2} \quad (4)$$

Now, equation (1) can be written as;

$$V_1 = i_{L_1} + \frac{di_{L_1}}{dt}$$

or, $$\frac{di_{L_1}}{dt} = V_1 - i_{L_1} \quad (5)$$

Equation (2) can be written as;

$$V_C = 2i_{L_2} + 2\left(\frac{di_{L_2}}{dt}\right) + V_2$$

or, $$\frac{di_{L_2}}{dt} = -i_{L_2} - \frac{V_2}{2} + \frac{V_C}{2} \quad (6)$$

Equation (3) can be written as;

$$\frac{dV_C}{dt} = \frac{i_C}{2} = \frac{i_{L_1} - i_{L_2}}{2} \quad (7)$$

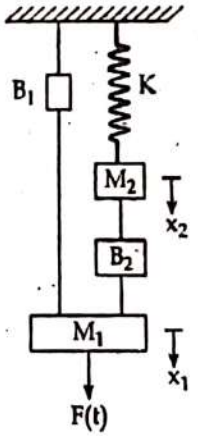
Hence, state space model for this system is;

$$\begin{bmatrix} \dot{i}_{L_1} \\ \dot{i}_{L_2} \\ \dot{V}_C \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} i_{L_1} \\ i_{L_2} \\ V_C \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$; which is in the form of $\dot{X} = AX + BU$.

The output equation is given by;

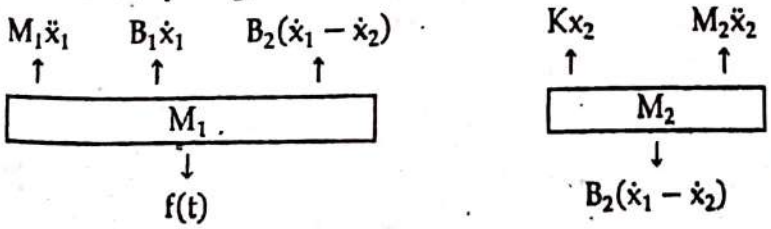
$$i_C = i_{L_1} - i_{L_2} = [1 \quad -1 \quad 0] \begin{bmatrix} i_{L_1} \\ i_{L_2} \\ V_C \end{bmatrix}$$; which is in the form of $Y = CX + DU$.

3. Derive the state variables model for the system shown below.



Solution:

The free body diagram for mass M_1 and mass M_2 is;



Now, for mass M_1; applying De-Alembert's principle,

Scanned with CamScanner

$$f(t) = M_1\ddot{x}_1 + B_2(\dot{x}_1 - \dot{x}_2) + B_1\dot{x}_1 = M_1\ddot{x}_1 + B_2\dot{x}_1 - B_2\dot{x}_2 + B_1\dot{x}_1 \quad (1)$$

For mass M_2; applying De-Alembert's principle,

$$Kx_2 + M_2\ddot{x}_2 = B_2(\dot{x}_1 - \dot{x}_2) = B_2\dot{x}_1 - B_2\dot{x}_2 \quad (2)$$

Both the equation is the order of two, so we need four state variables two for each.

Let, $x_1 = z_1$

$\dot{x}_1 = \dot{z}_1 = z_2$

and, $x_2 = z_3$

$\dot{x}_2 = \dot{z}_3 = z_4$

Now, equation (1) becomes;

$$f(t) = M_1\dot{z}_2 + B_2z_2 - B_2z_4 + B_1z_2 = M_1\dot{z}_2 + (B_1 + B_2)z_2 - B_2z_4 \quad (3)$$

or, $$\dot{z}_2 = \frac{f(t)}{M_1} - \left(\frac{B_1 + B_2}{M_1}\right)z_2 + \left(\frac{B_2}{M_2}\right)z_4$$

and, equation (2) becomes;

$$Kx_2 + M_2\ddot{x}_2 = B_2\dot{x}_1 - B_2\dot{x}_2$$

or, $$Kz_3 + M_2\dot{z}_4 = B_2z_2 - B_2z_4 \quad (4)$$

$$\therefore \quad \dot{z}_4 = -\left(\frac{K}{M_2}\right)z_3 + \left(\frac{B_2}{M_2}\right)z_2 - \left(\frac{B_2}{M_2}\right)z_4$$

The state space model for the system is;

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\\ \dot{z}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & -\left(\frac{B_1 + B_2}{M_1}\right) & 0 & \left(\frac{B_2}{M_1}\right)\\ 0 & 0 & 0 & 1\\ 0 & \left(\frac{B_2}{M_2}\right) & -\left(\frac{K}{M_2}\right) & -\left(\frac{B_2}{M_2}\right)\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\\ z_4\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{M_1}\\ 0\\ 0\end{bmatrix}f(t)$$

The displacement x_1 and x_2 act as an output. Therefore,

$x_1 = z_1$

$x_2 = z_3$

$$\therefore \quad \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}z_1\\ z_3\end{bmatrix}$$

The input equation is in the form of $\dot{X} = AX + BU$.

and, output equation is in the form of $Y = CX + DU$.

4. Deduce the state space model of transfer function $G(S) = \frac{S^2 + 1}{S^3 + 5S^2 - 2S + 1}$. **[2070 Ashadh]**

Solution:

Transfer function contains zero; so,

$$G(S) = \frac{Y(S)}{U(S)} = \frac{X_1(S)}{U(S)} \times \frac{Y(S)}{X_1(S)} = \frac{1}{S^3 + 5S^2 - 2S + 1} \times (S^2 + 1)$$

Comparing; we get,

$$\frac{X_1(S)}{U(S)} = \frac{1}{S^3 + 5S^2 - 2S + 1}$$

or, $$(S^3 + 5S^2 - 2S + 1)X_1(S) = U(S)$$

Taking inverse Laplace transform; we have,

$$\dddot{x}_1(t) + 5\ddot{x}_1(t) - 2\dot{x}_1(t) + x_1(t) = U(t)$$

or, $$\dddot{x}_1(t) = U(t) - x_1(t) + 2\dot{x}_1(t) - 5\ddot{x}_1(t)$$

Scanned with CamScanner

Let, $x_1 = z_1$

$\dot{x}_1 = \dot{z}_1 = z_2$

$\ddot{x}_1 = \ddot{z}_1 = \dot{z}_2 = z_3$

$\dddot{x}_1(t) = U(t) - x_1(t) + 2\dot{x}_1(t) - 5\ddot{x}_1(t) = U(t) - z_1 + 2z_2 - 5z_3$

so, $\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & 2 & -5\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u(t)$; which is input equation and takes the form $\dot{X} = AX + BU$.

Again, comparing; we have,

$$\frac{Y(S)}{X_1(S)} = S^2 + 1$$

or, $Y(S) = S^2X_1(S) + X_1(S)$

Taking inverse Laplace transform; we have,

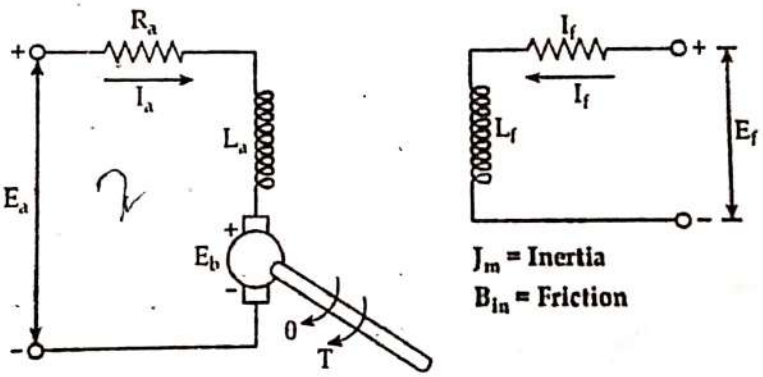
$y(t) = \ddot{x}_1(t) + x_1(t) = z_3 + z_1$

Therefore, state model becomes;

$$Y = [1 \quad 0 \quad 1]\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}$$

This is output equation and takes the form $Y = CX + DU$.

5. Develop state equation for motor circuit at below: **[2071 Chaitra]**



Solution:

We know that;

i) Electrical equation;

$$E_a = I_aR_a + L_a\frac{dI_a}{dt} + E_b$$

and, E_b = Back e.m.f. $= K_b\left(\frac{d\theta}{dt}\right)$

so, $E_a = I_aR_a + L_a\dot{I}_a + K_b\dot{\theta}$ (1)

where, E_a is the armature voltage.

I_a is the armature current.

Q is the angular displacement.

R_a is the armature resistance.

K_b is constant.

L_a is the armature inductance.

Scanned with CamScanner

Mechanical equation is;

$$T = J\frac{d^2\theta}{dt^2} + B\left(\frac{d\theta}{dt}\right) \quad (2)$$

or, $KI_a = J\ddot{\theta} + B\dot{\theta}$

Now, we have to choose state variables. For equation (1);

Let, $I_a = x_1$

$\dot{I}_a = \dot{x}_1$

Again for equation (2); we have,

$\theta = x_2$

$\dot{\theta} = \dot{x}_2 = x_3$

Now, equation (1) becomes; we have,

$$E_a = x_1R_a + L_a\dot{x}_1 + K_bx_3 \quad (3)$$

or, $\dot{x}_1 = \frac{E_a}{L_a} - \left(\frac{R_a}{L_a}\right)x_1 - \left(\frac{K_b}{L_a}\right)x_3$

Now, equation (2) becomes; we have,

$KI_a = J\ddot{\theta} + B\dot{\theta}$

$Kx_1 = J\ddot{\theta} + Bx_3$

$Kx_1 = J\dot{x}_3 + Bx_3$ (4)

$$\dot{x}_3 = -\left(\frac{B}{J}\right)x_3 + \left(\frac{K}{J}\right)x_1 \quad (5)$$

and, $\dot{x}_2 = x_3$

So, state space model becomes; we have,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-\left(\frac{R_a}{L_a}\right) & 0 & -\left(\frac{K_b}{L_a}\right)\\ 0 & 0 & 1\\ \left(\frac{K}{J}\right) & 0 & -\left(\frac{B}{J}\right)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + E_a\begin{bmatrix}\frac{1}{L_a}\\ 0\\ 0\end{bmatrix}$$

and, angular displacement θ is the output. So, output equation is;

$\theta = x_2$

$$\therefore \quad y = \theta = x_2 = [0 \quad 1 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

The input equation is in the form of $\dot{X} = AX + BU$ and output is in the form of $Y = CX + DU$.

6. The differential questions related to a system are $\frac{dx_1}{dt} = -3x_1 + x_2$ and $\frac{dx_2}{dt} = -2x_1 + u$ for $t > 0$. Its output is given by $y = x_2$. Derive the transfer function of the system with these differential equation and output equation. **[2073 Shrawan]**

Solution:

The equations are $\dot{x}_1 = -3x_1 + x_2$ and $\dot{x}_2 = -2x_1 + u$.

Writing in the form of matrix;

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-3 & 1\\ -2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

The output equation is;

Scanned with CamScanner

$$y = [0 \quad 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Comparing with $\dot{X} = AX + BU$ and $Y = CX + DU$; we get,

$$A = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [0 \quad 1] \text{ and } D = [0]$$

$$\text{Transfer function (T.F.)} = G(S) = \frac{Y(S)}{U(S)} = C^T[SI - A]^{-1}B + D$$

so, $$[SI - A] = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} S+3 & -1 \\ 2 & S \end{bmatrix}$$

$$|SI - A| = S(S+3) + 2 = S^2 + 3S + 2$$

Now,

$$[SI - A]^{-1} = \frac{\text{Adjoint of } |SI - A|}{|SI - A|} = \frac{1}{S^2 + 3S + 2}\begin{bmatrix} S & 1 \\ -2 & S+3 \end{bmatrix}$$

Again,

$$C = C^T = [0 \quad 1]$$

$$\therefore \quad \text{Transfer function (T.F.)} = \frac{Y(S)}{U(S)} = G(S) = [0 \quad 1] \times \frac{1}{S^2 + 3S + 2} \times \begin{bmatrix} S & 1 \\ -2 & S+3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= [0 \quad 1] \times \frac{1}{S^2 + 3S + 2} \times \begin{bmatrix} 1 \\ S+3 \end{bmatrix}$$

$$= (S+3)\left(\frac{1}{S^2 + 3S + 2}\right)$$

$$= \frac{S+3}{S^2 + 3S + 2}$$

Scanned with CamScanner

About The Author



Er. Umesh Kanta Ghimire (UKG) completed Bachelors in Electronics and Communication Engineering from Tribhuvan University, IOE, Thapathali Campus and running Masters in Technology and Innovation Management at Tribhuvan University, Pulchowk Campus.

Er. Ghimire is involving in teaching different engineering colleges and institutes as a part time lecturer on the electronics related subjects. Er. Ghimire has about three years of experience as a part time lecturer in IOE, Thapathali Campus. He had handled both theory and practical classes of several courses such as Basic Electronics Engineering, Computer Network, Control System, Remote Sensing, Engineering Material, Energy Environment and Society etc. and supervised some final years project too.

Our Publication's



G.L. Book House

Scanned with CamScanner